IMPACT OF DEMOGRAPHIC CHANGE ON INDUSTRY STRUCTURE IN AUSTRALIA

A joint study by the Australian Bureau of Statistics, the Department of Employment and Industrial Relations,
the Department of Environment, Housing and Community Development, the Department of Industry
and Commerce and the Industries Assistance Commission

MULTIPERIOD PLANNING MODELS AND GENERAL EQUILIBRIUM

by

Tony Meagher
Monash University

Preliminary Working Paper No. SP-09 Melbourne March 1977

The views expressed in this paper do not necessarily reflect the opinions of the participating agencies, nor of the Australian government.

IMPACT Project, Industries Assistance Commission, 606 St. Kilda Road, Melbourne, Victoria 3004 Australia
Area Code 03 Telephone 51 8611

CONTENTS

MULTIPERIOD PLANNING MODELS
AND GENERAL EQUILIBRIUM

by

Tony Meagher*
Monash University

1. INTRODUCTION

In his recent survey of multisector development planning models, Taylor (1975) noted

> "the ability of a linear programming model to simulate a general equilibrium or competitive resource allocation, complete with the prices from the dual solution."

However, he concluded that

> "the similarity of the model solution to a Walrasian equilibrium is often very faint, particularly on the price side. ... In short, present day models do not approach a Walrasian competitive resource allocation ... [and the interpretation of dual prices] as reflecting an economy in full general equilibrium easily breaks down."

Taylor's remarks were made in relation to a single period programming model, but we shall examine their relevance for a multiperiod model.

Traditionally, multisector planning models are expressed in terms of relationships between real economic variables. One important class of these models is formulated as a constrained optimization or programming problem in which some measure of consumption is maximized within limits imposed by technology and

---

* I am grateful to Peter Dixon for comments on earlier versions of this paper.

the availability of the factors of production. As is well known, a solution to a constrained optimization problem must satisfy the Kuhn-Tucker conditions, which involve not only the real variables but also a set of associated dual variables. Often the conditions on the dual variables can be interpreted as conditions for competitive market equilibrium, with the dual variables themselves playing the role of prices. However, as Taylor points out, apparently sensible programming problems expressed in real terms sometimes produce obscure relationships between the dual variables.

A statement of the Kuhn-Tucker conditions for the solution of a programming problem is a mathematically equivalent way of formulating the model represented by the original problem. This suggests an alternative approach in which the model is formulated in the first instance as a set of conditions defining a general equilibrium. These conditions can then be reinterpreted as Kuhn-Tucker conditions to determine the associated programming problem. A solution of this programming problem will undoubtedly describe a competitive general equilibrium, although the meaning of the problem itself may now be obscure.

We are interested in the relationship between multiperiod programming models and general equilibrium. The particular model we consider is a generalization of the "Dynamic Planning Model with Nonshiftable Capital Stocks by Sector" described by Taylor, and may be regarded as representing the elementary framework for models of that class.

Following tradition, the model is introduced in section 2 as a programming problem. In section 3 we show that the Kuhn-Tucker conditions for this problem can indeed be interpreted as conditions for multiperiod competitive equilibrium. Our method is to consider the activities of three groups of economic agents - producers, investors and consumers - operating independently in competitive markets. Adopting reasonable behavioural assumptions, optimization problems are formulated for each group, and the resulting optimization conditions are revealed to coincide with the Kuhn-Tucker conditions.

However, the result of section 3 is strained to the extent that it requires consumers to be remarkably prophetic with respect to future incomes and prices; in order to decide what to consume, they need to know in advance their net income in every period of the plan, and the price of every commodity in every period. In section 4 we suggest a modification which removes this unlikely information requirement. The consumers' average propensity to save is made exogenous to the model, rather than their rate of time preference as in the original version. In other words, the modification amounts to imposing a savings constraint on the model. The new version represents a multiperiod equilibrium in which no economic agent needs information on prices and incomes more than one period ahead. The technique for including the savings constraint becomes comparatively transparent when one begins with a general equilibrium formulation instead of a programming problem.

We conclude that multiperiod programming models are closely related to models of general competitive equilibrium, at least in their elementary form. Furthermore, our analysis suggests that there are advantages to be had by formulating such models as general equilibria in the first place.

2. THE MODEL AS A MULTIPERIOD PROGRAMMING PROBLEM[1]

When viewed as a programming problem, the model consists of an objective function and a set of constraints governing production, distribution and accumulation. Each of these aspects of the model will be discussed in turn.

(a) Production

As with most multisector planning models, the central element in this model is a fixed coefficient production function, embodying the assumptions of constant returns to scale and no substitution between inputs. In general, the production function for period t is written as

$$x_{jt} = \min \left\{ \frac{\kappa_{1jt}}{b_{1j}}, \ldots, \frac{\kappa_{mjt}}{b_{mj}}, \frac{L_{jt}}{l_j}, \frac{\alpha_{1jt}}{a_{1j}}, \ldots, \frac{\alpha_{njt}}{a_{nj}} \right\} \tag{2.1}$$

where x_{jt} = output of sector j,

κ_{ijt} = use of capital of type i in sector j,

L_{jt} = use of labour in sector j,

α_{ijt} = use of intermediate input of type i in sector j,

b_{ij} = amount of capital of type i required per unit output of sector j,

l_j = amount of labour required per unit output of sector j,

a_{ij} = amount of intermediate input of type i required per unit output in sector j,

m = number of types of capital

and n = number of types of intermediate input, assumed equal to the number of sectors.

---

[1] Although the model of this section is taken from Taylor (1975), the discussion is based on Kendrick (1970, 1972).

Since the output of sector j must be less than or equal to each of the arguments on the right hand side of the function (2.1), and since the input coefficients b_{ij}, a_{ij} and l_j are all non-negative by definition, we may rewrite (2.1) as

$$b_{ij}\, x_{jt} \leqq \kappa_{ijt} \qquad (i = 1,m;\ \ j = 1,n) \tag{2.2}$$

$$l_j\, x_{jt} \leqq L_{jt} \qquad (j = 1,n) \tag{2.3}$$

$$a_{ij}\, x_{jt} \leqq \alpha_{ijt} \qquad (i = 1,n;\ j = 1,n)\ . \tag{2.4}$$

Now the intermediate inputs of each type used in all sectors cannot exceed the total of each type available, so we require

$$\sum_{j=1}^{n} \alpha_{ijt} \leqq \alpha_{it} \qquad (i = 1,n) \tag{2.5}$$

where α_{it} = total amount of intermediate inputs of type i available in period t.

Similarly, if L_t is the total amount of labour available in period t, we require

$$\sum_{j=1}^{n} L_{jt} \leqq L_t. \tag{2.6}$$

Hence, summing (2.3) and (2.4) over sectors and adopting matrix notation, we obtain

$$\underline{l}'\underline{x}_t \leqq L_t \tag{2.7}$$

$$A\, \underline{x}_t \leqq \underline{\alpha}_t\ . \tag{2.8}$$

In the relation (2.2), output in each sector is constrained by each type of capital that is in place in that sector; we assume that once a capital good has been installed in a particular sector it cannot be subsequently transferred for use in a different sector.

When capital is not transferable, it can be conveniently measured in terms of the fixed bundle of capital goods ($b_{1j}, \ldots, b_{mj}$) required for unit production in each sector j. In other words, we measure the amount of capital k_{jt} in sector j in period t by the maximum output that can be produced from the capital installed in that sector. In terms of our previous notation, we have from (2.2),

$$k_{jt} = \min_i \{ \kappa_{ijt}/b_{ij} \} ,$$

and we require

$$(2.9) \qquad \underline{x}_t \leqq \underline{k}_t$$

in each period t. The relations (2.7) - (2.9) describe the production possibilities for the economy.

(b) Distribution

Besides the production function, the other fundamental element of a multisector planning model is the distribution relationship or materials balance constraint. This requires that the total amount of each commodity used for intermediate and final demand must not exceed the amount currently produced :

$$(2.10) \qquad \underline{x}_t \leqq \underline{\alpha}_t + \underline{d}_t$$

where $\underline{d}_t$ = final demand in period t. The left hand side of (2.10) implies that there is no joint production in the economy, but the assumption is one of convenience and a joint production matrix may be readily introduced if desired.

For our present purposes, we assume the final demand vector takes the simple form :

(2.11) $\underline{d}_t = \underline{c}_t + \underline{h}_t$

where $\underline{c}_t$ = private consumption in period t,

and $\underline{h}_t$ = investment in fixed capital in period t.

(c) Accumulation

In dynamic multisector models, investment demand is typically determined through relationships of the kind :

(2.12) $\underline{k}_{t+1} = \underline{k}_t + \underline{i}_t$

(2.13) $\underline{h}_t = B\underline{i}_t$

where $\underline{i}_t$ = increase in capital stocks in period t.

The first of these relationships describes the process of capital accumulation ; the capital available for production in any period is equal to the capital available in the previous period, plus the new capital created from investment goods in the previous period. Thus no part of current output of investment goods can contribute to production until the next period, but it can all contribute from that period on. This amounts to assuming that the length of the gestation lag is equal to the length of the time period.

The second relationship is a stock-flow relationship tying the increase in sectoral capital to the input of commodities to investment. If there is no excess capacity in the economy, the relations (2.9), (2.12) and (2.13) can be combined to give

$$\underline{h}_t = B(\underline{x}_{t+1} - \underline{x}_t) .$$

Hence the investment theory underlying (2.12) and (2.13) is a multisector version of the acceleration principle, with current demands for investment goods depending on future growth of output.

The matrix B in (2.13) is a matrix of marginal capital coefficients and is not strictly the same as the matrix of average coefficients in (2.2). However, it is usual to assume that the marginal and average coefficients are equal and we do so here. Both this assumption and the assumption about the length of the gestation lags are substantive, but the quality of the data available for estimating capital coefficients and gestation lags does not generally support the more refined formulation.

Note also that the row index of the B matrix has been increased to span all sectors rather than just those producing capital goods; hence B will typically contain numerous zero rows.

Drawing together the relationships describing production possibilities, distribution and accumulation, we have the following basic structure for a multiperiod model :

(2.14) $\underline{x}_t \leqq A\underline{x}_t + \underline{c}_t + B\underline{i}_t$

(2.15) $\underline{1}'\underline{x}_t \leqq L_t$

(2.16) $\underline{x}_t \leqq \underline{k}_t$

(2.17) $\underline{k}_{t+1} = \underline{k}_t + \underline{i}_t$.

(d) Objective Function

The conditions (2.14) - (2.17) only define the set of feasible growth paths for the economy; to discriminate between the feasible paths we must also define a planning horizon and an objective function. A typical choice for the latter is the discounted utility of consumption during the plan, plus the value of capital stocks at the end of the plan.[2]

---

[2] Determining suitable terminal conditions, in our case the value to be placed on terminal capital stocks, presents problems. Taylor (1975) discusses some of the methods that have been used in the literature.

In general, for a T-period planning horizon, utility will be given by a multiperiod function depending on the consumption of each commodity in each of the T time periods. However, we assume that utility in one period is not directly dependent on consumption or utility in any other period, and that utilities can be added once they have been discounted to allow for the time preference of society. Then the multiperiod utility function can be expressed as the weighted sum of the single period utility functions $U(\underline{c}_t)$, which we assume to be strictly concave[3] with positive first derivatives.

If ρ is the rate of time preference, the discount factor

$$\varepsilon_t = (1+\rho)^{-t}$$

defines the rate at which utility in period t transforms into base period utility, and

$$\sum_{t=1}^{T} \varepsilon_t U(\underline{c}_t)$$

gives the utility of the consumption stream $\{\underline{c}_1, \ldots, \underline{c}_T\}$ measured in base period utility units. Let $\bar{\underline{X}}_{T+1}$ be the valuation (per unit) placed on terminal capital stocks, also measured in base period utility units.

---

[3] The conventional assumption of strict quasiconcavity would be sufficient to ensure a unique solution to the fixed weight problem (2.18) considered in this section. But it is convenient to assume strict concavity for later work in which the weights ε_t are varied - see footnote 9. In any case, the stronger mathematical assumption is not more restrictive economically because of the following result : any observation of consumer behaviour that invalidates the hypothesis that consumers maximize a strictly concave utility function must also invalidate the hypothesis that they maximize any utility function. In other words, if we assume that consumers maximize utility at all, we can assume that their utility function is strictly concave. See Dixon (1975), p.96. The result also holds for the utility maximizing society considered in the present context.

Then the optimal growth path is given by the solution of the programming problem[4,5] :

$$(2.18) \quad \text{maximize} \quad \sum_{t=1}^{T} \varepsilon_t U(\underline{c}_t) + \bar{\underline{\lambda}}'_{T+1}\, \underline{k}_{T+1}$$

$$\begin{aligned}
\text{subject to} \quad -(I-A)\underline{x}_t + \underline{c}_t + B\underline{i}_t &\leqq \underline{0} & \quad (t = 1,T)\\
\underline{1}'\underline{x}_t &\leqq L_t & \text{"}\\
\underline{x}_t &\leqq \underline{k}_t & \text{"}\\
\underline{k}_1 &= \bar{\underline{k}}_1 & \text{"}\\
\underline{k}_{t+1} &= \underline{k}_t + \underline{i}_t & \text{"}\\
\underline{x}_t,\ \underline{c}_t,\ \underline{i}_t,\ \underline{k}_t &\geqq \underline{0} & \text{"}\\
\underline{k}_{T+1} &\geqq \underline{0} &
\end{aligned}$$

where $\bar{\underline{k}}_1$ = given initial capital stocks.

We have now assembled, in (2.18), a representative multiperiod programming model. In the next section we take the conditions for a solution to this model and use them to define a multiperiod general equilibrium.

---

[4] For all the programming problems considered in this paper we assume that the required constraint qualifications are satisfied. For a full discussion of constraint qualifications, see Mangasarian (1969), ch. 7.

[5] The objective function here is actually a generalization of that cited by Taylor. To obtain the latter, we assume the utility function has the Leontief form :

$$U(\underline{c}_t) = \min \left\{\frac{c_{1t}}{c_1}, \ldots, \frac{c_{nt}}{c_n}\right\} = \Gamma_t$$

where $\underline{c}$ is an exogenously determined bundle of consumption goods. We may then replace $\underline{c}_t$ with $\Gamma_t\underline{c}$ in the materials balance constraint.

3. THE MODEL AS A MULTIPERIOD GENERAL EQUILIBRIUM

We define a multiperiod general equilibrium as the set of non-negative vectors and scalars

$$\Omega = [\{\underline{c}_t,\ \underline{x}_t,\ \underline{i}_t,\ \underline{k}_t,\ \underline{\phi}_t,\ \underline{\chi}_t,\ \underline{\Pi}_t,\ w_t \mid t = 1,T\},\ \underline{k}_{T+1},\ \underline{\chi}_{T+1}] \tag{3.1}$$

satisfying the following conditions :

demand and supply conditions

commodities

$$A\underline{x}_t + \underline{c}_t + B\underline{i}_t \leqq \underline{x}_t \qquad (t = 1,T) \tag{3.2}$$

labour

$$\underline{1}'\underline{x}_t \leqq L_t \qquad \text{"} \tag{3.3}$$

capital

$$\underline{x}_t \leqq \underline{k}_t \qquad \text{"} \tag{3.4}$$

capital accumulation conditions

$$\underline{k}_1 = \bar{\underline{k}}_1 \tag{3.5}$$

$$\underline{k}_{t+1} = \underline{k}_t + \underline{i}_t \qquad \text{"} \tag{3.6}$$

market clearing conditions

commodities

$$\underline{\phi}_t{}' (\underline{x}_t - A\underline{x}_t - \underline{c}_t - B\underline{i}_t) = 0 \qquad \text{"} \tag{3.7}$$

labour

$$w_t (\underline{1}'\underline{x}_t - L_t) = 0 \qquad \text{"} \tag{3.8}$$

capital

$$\underline{\Pi}_t{}' (\underline{x}_t - \underline{k}_t) = 0 \qquad \text{"} \tag{3.9}$$

profit conditions

producers

$$\underline{\phi}_t{}' \leqq \underline{\phi}_t{}'A + w_t\underline{1}' + \underline{\Pi}_t{}' \qquad \text{"} \tag{3.10}$$

$$(\underline{\phi}_t{}' - \underline{\phi}_t{}'A - w_t\underline{1}' - \underline{\Pi}_t{}')\, \underline{x}_t = 0 \qquad \text{"} \tag{3.11}$$

investors

(3.12) $\underline{\chi}'_{t+1} \leqq \underline{\phi}_t{}'B$ (t = 1,T)

(3.13) $(\underline{\chi}'_{t+1} - \underline{\phi}_t{}'B)\underline{i}_t = 0$ "

(3.14) $\underline{\Pi}_t = \underline{\chi}_t - \underline{\chi}_{t+1}$ "

consumption conditions

(3.15) $\varepsilon_t \nabla U \leqq \underline{\phi}_t{}'$ "

(3.16) $(\varepsilon_t \nabla U - \underline{\phi}_t{}')\underline{c}_t = 0$ "

terminal conditions

(3.17) $\underline{\chi}_{T+1} = \bar{\underline{\chi}}_{T+1}$.

The equilibrium is defined over the time period t = 1,T.

The endogenous variables of the model are :

$\underline{c}_t$ = commodities consumed,

$\underline{x}_t$ = commodities produced,

$\underline{i}_t$ = increases in capital stocks,

$\underline{k}_t$ = capital stocks,

$\underline{\phi}_t$ = commodity prices,

$\underline{\chi}_t$ = capital prices,

$\underline{\Pi}_t$ = rental rates for capital,

w_t = wage rate,

all defined for each period t = 1,T, and

$\underline{k}_{T+1}$ = terminal capital stocks,

$\underline{\chi}_{T+1}$ = terminal capital prices.

The exogenous variables are :

L_t = labour supply in period t,

$\bar{\underline{k}}_1$ = initial capital stocks,

$\bar{\underline{\chi}}_{T+1}$ = terminal capital values.

The parameters are :

A = intermediate input coefficients,

B = capital coefficients,

$\underline{l}$ = labour input coefficients,

ε_t = discount factor for converting utility into base period units,

U = consumers' utility function.

The conditions (3.7), (3.8), (3.9), (3.11) and (3.13) together imply

$$(3.18) \qquad \underline{\phi}_t{}'\underline{c}_t + \underline{\chi}'_{t+1}\,\underline{i}_t = w_t L_t + \underline{\Pi}_t{}'\underline{k}_t,$$

which is a statement of Walras' Law for the model. Also, (3.6) and (3.14) imply

$$(3.19) \qquad \underline{\chi}'_1\underline{k}_1 + \underline{\chi}'_{T+1}\,\underline{k}_{T+1} = \sum_{t=1}^{T} (\underline{\Pi}_t{}'\underline{k}_t - \underline{\chi}'_{t+1}\,\underline{i}_t).$$

All the conditions are homogeneous in incomes and prices with the exception of the consumption conditions, which define the numeraire. As we shall see, the marginal utility of consumption expenditure has been arbitrarily set to unity.

Each of the pairs of conditions (3.2) and (3.7), (3.4) and (3.9), (3.10) and (3.11), (3.12) and (3.13), and (3.15) and (3.16) is equivalent to nT equations, where n is the number of commodities. The pair (3.3) and (3.8) is equivalent to T equations, and (3.5), (3.6), (3.14) and (3.17) contribute 2nT + 2n, making 7nT + T + 2n equations in all. This is the same as the number of endogenous variables in the model. While the equality of the number of endogenous variables and the number of equations is neither necessary nor sufficient for the existence of a solution, it is nevertheless true that in the majority of well behaved and interpretable economic models, this condition is met.

Returning briefly to the programming problem (2.18), we note that the set of non-negative vectors

$$\Omega_P = [\{\underline{c}_t, \underline{x}_t, \underline{i}_t, \underline{k}_t \mid t = 1,T\}, \underline{k}_{T+1}]$$

is a solution to that problem if and only if there also exists a set of non-negative vectors and scalars

$$\Omega_D = [\{\underline{\phi}_t, \chi_t, \underline{\Pi}_t, w_t \mid t = 1,T\}, \chi_{T+1}]$$

such that the conditions (3.2) - (3.17) are satisfied. In purely mathematical terms, therefore, the programming model and the general equilibrium are equivalent. The analysis of this section will indicate the economic interpretation that must be placed on the programming model if it is to be regarded as defining a general equilibrium.

Of the conditions which define the equilibrium,(3.2) - (3.6) are expressed in physical terms while the others are in monetary terms. The former group, consisting of demand and supply and capital accumulation conditions, represent the constraints imposed on equilibrium by technology and by the availability of the factors of production. They are identical with the constraints of the programming model (2.18), and have already been discussed in that context. The interpretation of the second group requires some further assumptions about the organization of the economy and the behaviour of the economic agents.

We assume that the economy is populated by three groups of economic agents, namely consumers, producers and investors, who operate in an environment of competitive commodity and factor markets. The first day of each period is a marketing day, during which contracts are let and payments are made for the supply of commodities and factors. In the remainder of the period, the contracted supplies are delivered and the physical operations of production and consumption take place.

At the end of the base period each agent is assumed to formulate a plan which governs his economic behaviour over the following T periods. The agents' plans will depend on their expectations of future movements of prices, incomes and interest rates. We assume that all expectations are subsequently realized so no agent will have cause to change his initial plan during the planning period and we need draw no distinction between expected and actual values of the monetary variables. Furthermore, since interest rates do not appear explicitly in the model, the monetary variables should be interpreted as having been discounted back to the base period a priori.

Equilibrium conditions for the economy are determined according to the various behavioural assumptions made about the economic agents, as we shall now establish.

(a) Market Clearing Conditions

Markets exist for commodities, labour and capital. The corresponding demand and supply conditions (3.2) - (3.4) prohibit excess demand in any market at equilibrium but allow excess supply. We assume that suppliers of factors and commodities respond to excess supply by lowering their prices. It follows that the only price which is consistent with both excess supply and equilibrium is a price of zero. Hence the market clearing conditions (3.7) - (3.9) are included to ensure that, if any commodity or factor is in excess supply at equilibrium, the commodity or factor is a free good.

(b) Producers

Consider a producer of commodity j. To produce an output x_j, he requires amounts $a_{ij}x_j$ of each intermediate input i, l_jx_j of labour and x_j of capital. These input requirements do not vary with the level of output and are the same for every producer of commodity j. Furthermore, the prices paid for factors and received for output are also the same for all producers of the commodity. Hence with no loss of generality, we may consider that there is only one producer of each commodity.

The behavioural assumption for producers is that they choose the combination of inputs and outputs over the planning period which maximizes their profits. All inputs are variable when the production plan is formulated, so producers will not purchase inputs in excess of those actually required for production. Hence, for the j^{th} sector, the production plan is the solution of the programming problem :

$$(3.20) \quad \text{maximize} \quad \sum_{t=1}^{T} (\phi_{jt}x_{jt} - \sum_{i=1}^{n} \phi_{it}a_{ij}x_{jt} - w_t l_j x_{jt} - \Pi_{jt}x_{jt})$$

$$\text{subject to} \quad x_{jt} \geqq 0 \qquad (t = 1,T)$$

where ϕ_{jt} = market price of commodity j,

w_t = wage rate,

and Π_{jt} = rental rate for capital in sector j in period t.

In a formal way, we can write down the conditions for a solution to this problem as follows :

(3.21) $\phi_{jt} - \sum_{i=1}^{n} \phi_{it} a_{ij} - w_t l_j - \Pi_{jt} \leqq 0$ (t = 1,T)

(3.22) $(\phi_{jt} - \sum_{i=1}^{n} \phi_{it} a_{ij} - w_t l_j - \Pi_{jt}) x_{jt} = 0$ "

(3.23) $x_{jt} \geqq 0$ " .

These conditions require that the price of commodity j does not exceed the cost of the inputs used in its production. If its price is less than the cost of production, the commodity will not be produced.

Intuitively, positive profits must be excluded from equilibrium because linear technology and competitive markets lead to the situation where any level of profits is possible if some positive level is possible, at least from the point of view of a particular producer. Hence positive profits would imply infinite demand for factors of production which are only available in limited supply.

Note also that producers will be indifferent between various levels of output at the equilibrium set of prices, since any level of output yields zero profits. However, should they choose to produce a non-equilibrium level of output, one or other of the equilibrium conditions will be violated, thus upsetting the equilibrium set of prices and precipitating a new round of adjustments.

Comparing (3.10) - (3.11) with (3.21) - (3.22) reveals that the profit conditions on the production of commodities in the definition of the multiperiod equilibrium can be derived from a consideration of profit maximizing behaviour on the part of producers.

The programming problem (3.20) takes into account market prices for every period in the planning horizon. Yet none of the conditions (3.21) - (3.23) for its solution is intertemporal. It follows that the producer would have no reason to change his production plan if he maximized his profits period by period rather than over the whole T periods taken as a whole. In other words, in order to maximize profits over T periods, producers need to know only current market prices, given our assumptions about the technology and market structure of the economy.

(c) Investors

Now consider the r^th^ investor in the j^th^ sector. His initial holding of capital is given exogenously as $\bar{k}^r_{j1}$ where

$$\sum_r \bar{k}^r_{j1} = \bar{k}_{j1}$$

and $\bar{k}_{j1}$ = total initial capital in sector j.

On the first marketing day, the investor may sell all or part of his initial holding, or he may add to it by buying capital from other investors in the sector. When he has obtained his desired stock, he rents it to producers operating in the sector in the first period. His net purchase of capital on the first day is the difference between the amount k^r_{j1} that he rents and his initial holding $\bar{k}^r_{j1}$; his outlay is thus

$$\chi_{j1} (k^r_{j1} - \bar{k}^r_{j1})$$

where χ_{j1} = market price of capital on the first day. The total net purchases of all investors in the j^th^ sector must, of course, be zero.

In addition to buying or selling existing capital, each investor may let contracts to producers in various sectors to supply him with the investment goods required for new capital formation. Although payment for these goods is made on the first day, they are not delivered until later in the period, so new capital only becomes available for renting to producers in the second and subsequent periods. If the r^{th} investor intends to form i^r_{j1} units of new capital in the first period he must purchase $b_{ij}i^r_{j1}$ of each good i for a total outlay of

$$\sum_{i=1}^{n} \phi_{i1}b_{ij}i^r_{j1} .$$

Thus the net income for the r^{th} investor in sector j on operations in the first period is given by

$$\Pi_{j1}k^r_{j1} - \chi_{j1}(k^r_{j1} - \bar{k}^r_{j1}) - \sum_{i=1}^{n} \phi_{i1}b_{ij}i^r_{j1}$$

where Π_{j1} = market rental rate for capital on the first day.

In subsequent periods, the expression for net income is similar, the only difference being in the net purchase of existing capital. Since k^r_{jt-1} is the amount of capital rented during period t-1 and i^r_{jt-1} is the amount of new capital formed during that period,

$$k^r_{jt-1} + i^r_{jt-1}$$

is the r^{th} investor's holding of capital at the commencement of trading on the t^{th} marketing day. His holding at the end of the day is k^r_{jt}, the amount he rents during the t^{th} period. Hence his net purchase is

$$k^r_{jt} - k^r_{jt-1} - i^r_{jt-1} ,$$

and his net income in the t^{th} period is

$$\Pi_{jt}k^r_{jt} - \chi_{jt}(k^r_{jt} - k^r_{jt-1} - i^r_{jt-1}) - \sum_{i=1}^{n} \phi_{it}b_{ij}i^r_{jt} \qquad (t = 2,T).$$

As he is not planning to continue operations beyond the $(T+1)^{th}$ marketing day, the r^{th} investor will sell off his holding at the end of the plan, yielding an income of

$$\chi_{jT+1}\,(k^r_{jT} + i^r_{jT}).$$

The market price χ_{jT+1} of capital on the $(T+1)^{th}$ day is given exogenously by the terminal condition (3.17).

The behavioural assumption for investors is that they maximize their net income during the plan. Hence the investment plan for the r^{th} investor in the j^{th} sector is given by the solution of the programming problem :

$$(3.24) \quad \text{maximize} \quad \sum_{t=1}^{T} \Pi_{jt}k^r_{jt} - \chi_{j1}(k^r_{j1} - \bar{k}^r_{j1})$$
$$- \sum_{t=2}^{T} \chi_{jt}\,(k^r_{jt} - k^r_{jt-1} - i^r_{jt-1})$$
$$+ \chi_{jT+1}\,(k^r_{jT} + i^r_{jT})$$
$$- \sum_{t=1}^{T} \sum_{i=1}^{n} \phi_{it}b_{ij}i^r_{jt}$$

$$\text{subject to} \quad k^r_{jt} \geqq 0 \qquad (t = 1,T)$$
$$i^r_{jt} \geqq 0 \qquad \text{"} \quad .$$

The conditions for a solution to this problem are the following :

$$(3.25) \quad \Pi_{jt} - \chi_{jt} + \chi_{jt+1} \leqq 0 \qquad (t = 1,T)$$

$$(3.26) \quad (\Pi_{jt} - \chi_{jt} + \chi_{jt+1})\,k^r_{jt} = 0 \qquad \text{"}$$

$$(3.27) \quad \chi_{jt+1} - \sum_{i=1}^{n} \phi_{it}b_{ij} \leqq 0 \qquad \text{"}$$

$$(3.28) \quad (\chi_{jt+1} - \sum_{i=1}^{n} \phi_{it} b_{ij}) \, k^r_{jt} = 0 \qquad (t = 1,T)$$

$$(3.29) \quad k^r_{jt} \geqq 0 \qquad "$$

$$(3.30) \quad i^r_{jt} \geqq 0 \qquad " \quad .$$

From (3.25), it is not possible in equilibrium for an investor to make a profit by buying a unit of capital for χ_{jt} on the t^{th} marketing day, renting it to producers for Π_{jt} during the rest of the period t, and then selling it for χ_{jt+1} on the $(t+1)^{th}$ marketing day. At best he can break even on the transactions. If prices and rents are such that he would make a loss, then (3.26) indicates that he will not rent any capital in period t, i.e., he will sell all his capital on the t^{th} marketing day. Since the total amount of capital rented by all investors in sector j in period t must be equal to the amount in existence in the sector at that time, not all investors can refrain from renting, i.e., $k^r_{jt} > 0$ for some investor in every period.[6] Hence condition (3.25) must hold with strict equality in every period, giving the profit condition (3.14) of the multiperiod equilibrium.

The condition (3.27) excludes the possibility of investors making a positive profit from purchasing a bundle of investment goods b_{ij} (i = 1,n) costing

$$\sum_{i=1}^{n} \phi_{it} b_{ij}$$

in period t, forming them into a unit of new capital for sector j, and then selling the capital for χ_{jt+1} on the $(t+1)^{th}$ marketing day.

---

[6] Note that we do conceptually allow all investors to sell off their holdings of capital on the $(T+1)^{th}$ marketing day, but all questions concerning the operation of the economy beyond that time are subsumed in the specification of the terminal conditions (3.17).

Once again the best they can do is break even, and if they can only make losses, according to (3.28) they will not form any new capital. The profit condition (3.12) of the multiperiod equilibrium is derived directly from (3.27), and the condition (3.13) is derived from (3.28) by summing over investors.

As with the producer's maximization problem, the conditions (3.25) and (3.26) excluding positive profits are formally necessary to ensure that the objective function of the problem (3.24) is bounded. But they make good intuitive sense as equilibrium conditions for, in their absence, demand for investment goods would become infinite, resulting in excess demand for the limited factors of production.

Earlier in this section we noted that the equilibrium conditions (3.21) - (3.23) for producers can be obtained from a series of programming problems, one for each period, rather than from a single problem covering all T periods of the planning horizon. The same result holds here for investors. Suppose that on the t^{th} marketing day, the r^{th} investor in the j^{th} sector buys k^r_{jt} units of existing capital and orders sufficient investment goods to create i^r_{jt} units of new capital by the beginning of the next period. During the rest of period t he rents the k^r_{jt} units of capital he bought, and on the $(t+1)^{th}$ marketing day he sells his entire holding of $(k^r_{jt} + i^r_{jt})$ units. The maximum profit the investor can make from these transactions is given by the solution of the programming problem :

$$\text{(3.31)} \quad \text{maximize } \Pi_{jt}k^r_{jt} - \chi_{jt}k^r_{jt} - \sum_{i=1}^{n} \phi_{it}b_{ij}i^r_{jt} + \chi_{jt+1}(k^r_{jt} + i^r_{jt})$$

$$\text{subject to } k^r_{jt} \geqq 0$$

$$i^r_{jt} \geqq 0 \ .$$

Taking the conditions for a solution to (3.31) for each period $t = 1, T$ gives the complete set of conditions (3.25) - (3.30) for a solution to the multiperiod problem (3.24).

Hence through the conceptual device of selling all his holding of capital at the beginning of each period before entering into transactions for the next, the investor is able to separate his multiperiod profit maximizing problem into a series of single period problems. The information he needs to solve each of these problems consists of the price and rental rate for capital on the current marketing day, and the price of capital on the next.

(d) Consumers

We assume that the combined preferences of all consumers are described by a single utility function[7] which has the same properties as the social utility function described in the previous section. Hence, the utility of the consumption stream $\{\underline{c}_1 \ldots, \underline{c}_T\}$, measured in base year utility units, is

$$\sum_{t=1}^{T} \varepsilon_t U(\underline{c}_t)$$

where ε_t = discount factor for converting current utility into base period units,

and U = consumers' single period utility function.

Consumers choose the consumption plan which maximizes their utility subject to a budget constraint imposed by their total income y during the planning period, less some minimum level of savings $\bar{s}$ representing their provision for eventualities beyond the end of the plan. Hence the consumption plan is the solution of the programming problem :

---

[7] In an empirical study using U.S. data for 13 commodities and 3 "consumers" (poor, middle and rich), Dixon (1975) concluded that "... if each household behaves as if it maximizes an additive utility function subject to a budget constraint, then it is reasonable

$$(3.32) \quad \text{maximize} \quad \sum_{t=1}^{T} \varepsilon_t \, U(\underline{c}_t)$$

$$\text{subject to} \quad \sum_{t=1}^{T} \underline{\phi}_t' \underline{c}_t \leqq y - \bar{s}$$

$$\underline{c}_t \geqq \underline{0} \qquad (t = 1,T).$$

Necessary and sufficient conditions for a solution to this problem are the following :

$$(3.33) \quad \varepsilon_t \nabla U - \delta \underline{\phi}_t' \leqq \underline{0} \qquad (t = 1,T)$$

$$(3.34) \quad (\varepsilon_t \nabla U - \delta \underline{\phi}_t') \, \underline{c}_t = 0 \qquad \text{"}$$

$$(3.35) \quad \sum_{t=1}^{T} \underline{\phi}_t' \underline{c}_t - (y - \bar{s}) \leqq 0 \qquad \text{"}$$

$$(3.36) \quad \delta \Big[\sum_{t=1}^{T} \underline{\phi}_t' \underline{c}_t - (y - \bar{s}) \Big] = 0$$

$$(3.37) \quad \underline{c}_t \geqq 0 \qquad (t = 1,T)$$

$$(3.38) \quad \delta \geqq 0$$

where δ is the Lagrange multiplier on the budget constraint.

The optimality conditions require

$$(3.39) \quad \delta = \varepsilon_t \, (\partial U / \partial c_{jt}) / \phi_{jt}$$

whenever $c_{jt} > 0$, that is, the ratio of discounted marginal utility to price must be the same for every consumed commodity in every period. It follows that the Lagrange multiplier δ has the interpretation of the marginal utility of consumption expenditure. Furthermore, given our assumptions about the utility function, δ must be positive and the budget constraint must hold with strict equality.

The consumption conditions (3.15) and (3.16) of the multi-period equilibrium are the same as the conditions (3.33) and (3.34) for the maximization of consumers utility if we impose on the latter the additional requirement that $\delta = 1$. In other words, the consumption

---

to assume that aggregate consumption responds to price and expenditure changes as though the community maximizes an additive utility function subject to an aggregate budget constraint."

conditions incorporate the choice of the numeraire for the multiperiod equilibrium, namely the marginal utility of expenditure.

By definition, consumers' income in period t is given by

$$y_t = w_t L_t + \underline{\Pi}_t{}' \underline{k}_t$$

where $\sum_{t=1}^{T} y_t = y.$

Also, by definition,

$$y_t = s_t + \underline{\phi}_t{}' \underline{c}_t$$

where s_t = level of savings in period t,

and $\sum_{t=1}^{T} s_t = \bar{s}$.

Therefore, using the implicit condition (3.18),

$$s_t = \underline{\chi}'_{t+1} \, \underline{i}_t$$

in equilibrium. Thus a condition for equilibrium is that the level of savings chosen by consumers in each period is just sufficient to finance the investment stream which converts the initial capital stocks $\underline{k}_1$ into the terminal capital stocks $\underline{k}_{T+1}$. If this is not the case for any given set of prices, one or other of the equilibrium conditions (3.2) - (3.17) will be violated, and the market forces we have described will operate to produce a new set of prices.

Since the marginal utility of expenditure must be the same in every period, it is not possible for consumers to decide on consumption levels for any particular period without determining them for all other periods in the plan. Unlike producers and investors, consumers cannot decompose their multiperiod problem into separate single period problems. In order to formulate their plan they must know in advance their net income in every period and the prices of every commodity in every period. In the next section we shall show that this unlikely information requirement is removed when a savings constraint is included in the definition of equilibrium.

It is now clear that the programming model (2.18) defines a general equilibrium, albeit with a strong requirement about the information available to consumers, provided its objective function reflects consumers' preferences. In that case, the model simulates the autonomous operation of a market economy and the programming problem can be regarded as an algorithm for computing equilibrium values of the economic variables. The dual variables Ω_D can be identified as market prices.

When the objective function of the programming model reflects planners' prices, as would be appropriate in a planned economy, the solution of the model no longer represents a general equilibrium. However, the dual variables could still be employed by a planning agency to implement their plan without centralized direction for any primal variable in Ω_P except consumption. If prices of commodities and factors are set according to the dual solution of the model, and producers and investors are instructed to maximize profits, the production and accumulation plans would be achieved through the independent decisions of those agents. At the dual prices, consumers would not choose to buy the consumption goods produced and some rationing scheme for consumption goods would be required.

4. THE MODEL WITH A SAVINGS CONSTRAINT

When formulating the consumption plan, consumers have previously been supposed to budget for their entire income during the planning period. Here we adopt a different assumption which makes the formulation of their plan less exacting, namely that consumers save a fixed proportion s of their income in each period. Total savings during the plan is now determined by the amounts saved in each period, and is no longer the subject of an independent decision by consumers.

Formally, consumers will choose the commodities $\underline{c}_t$ in each period t which

(4.1) $\quad$ maximize $U(\underline{c}_t)$

$\quad\quad$ subject to $\underline{\phi}_t'\underline{c}_t = (1-s)y_t$

where income in period t is given identically by[8]

(4.2) $\quad y_t \equiv w_t L_t + \underline{\Pi}_t'\underline{k}_t$.

In this view, the preferred consumption stream $\{\underline{c}_1,\ldots,\underline{c}_T\}$ is deduced from a series of single period problems (4.1) rather than a multi-period problem, so consumers no longer require more than current information on prices and incomes.

The necessary and sufficient conditions for a solution to (4.1) are the following :

(4.3) $\quad \nabla U - \delta_t\underline{\phi}_t' < 0$

(4.4) $\quad (\nabla U - \delta_t\underline{\phi}_t')\underline{c}_t = 0$

(4.5) $\quad \underline{\phi}_t'\underline{c}_t - (1-s)y_t = 0$

(4.6) $\quad \underline{c}_t \geqq 0$

(4.7) $\quad \delta_t > 0$

where δ_t is the Lagrange multiplier on the budget constraint. The optimality conditions require

$$\delta_t = (\partial U/\partial c_{jt})/\phi_{jt}$$

whenever $c_{jt} > 0$; that is, the ratio of marginal utility to price must

---

[8] Note that this formulation can readily cope with different propensities to save from wages and profits.

be the same for every commodity consumed. As before, the Lagrange multiplier can be interpreted as the marginal utility of consumption expenditure, but utility is now measured in current rather than base period units, and there is no requirement that the marginal utility be the same in every period. If we were to assume that consumers still equate marginal utilities measured in base period units, the model would reveal the time preference pattern implied by their behaviour in saving a fixed proportion of their income in each period. This can be contrasted with the previous formulation in which time preference was specified exogenously and the model determined the average propensity to save in each period.

The consumption conditions (3.15) - (3.16) in the definition of the original multiperiod equilibrium are now replaced by the following :

$$(4.8) \qquad \nabla U - \delta_t \underline{\phi}_t' \leqq 0 \qquad (t = 1,T)$$

$$(4.9) \qquad (\nabla U - \delta_t \underline{\phi}_t')\underline{c}_t = 0 \qquad ''$$

$$(4.10) \qquad \underline{\phi}_t' \underline{c}_t = (1-s)(w_t L_t + \underline{\Pi}_t' \underline{k}_t) \qquad '' \quad .$$

We have introduced T new variables δ_t and effectively replaced nT equations with nT + T equations, so the number of variables and equations remain equal. The set of conditions (3.2) - (3.14), (4.8) - (4.10), (3.17) defines a modified multiperiod general equilibrium in which the economic agents are never required to possess information on incomes and prices more than one period ahead.

Consider again the programming problem (2.18) but suppose the factors ε_t in the objective function are simply exogenous weights bearing no necessary relation to a given rate of time preference.

The new role of the weights will become apparent shortly. As we indicated in the previous section, the set of non-negative vectors

$$\Omega_P = [\{\underline{c}_t, \underline{x}_t, \underline{i}_t, \underline{k}_t \mid t = 1,T\}, \underline{k}_{T+1}]$$

is a solution to that problem if and only if there also exists a set of non-negative vectors and scalars

$$\Omega_D = [\{\underline{\phi}_t, \underline{\chi}_t, \underline{\Pi}_t, w_t \mid t = 1,T\}, \underline{\chi}_{T+1}]$$

such that conditions (3.2) - (3.17) are satisfied. If, in addition,

$$(4.11) \quad \varepsilon_t = (1-s)(w_t L_t + \underline{\Pi}_t{}' \, \underline{k}_t) \, / \, \nabla U . \underline{c}_t \qquad (t = 1,T),$$

then $\Omega = (\Omega_P, \Omega_D)$ also satisfies (4.8) - (4.10), and hence is an instance of the modified general equilibrium. Comparing (4.11) with (4.9) and (4.10) indicates that, in equilibrium, the weights ε_t will be equal to the reciprocal of the marginal utility of expenditure, measured in current utility units, i.e.,

$$(4.12) \quad \varepsilon_t = \delta_t^{-1}.$$

We may continue to interpret them as the discount factors which convert current utility into base period utility, but now their determination is endogenous.

To compute the modified general equilibrium, we solve a sequence of problems of the form (2.18), systematically changing the values of the weights ε_t from problem to problem.[9] In particular, if $v(\nu)$ denotes the value of an endogeous variable v obtained in the ν^{th} solution to the programming problem, we set the weights for the $(\nu+1)^{th}$ problem according to

$$(4.13) \quad \varepsilon_t(\nu+1) = [1-s][w_t(\nu)L_t + \underline{\Pi}_t{}'(\nu)\underline{k}_t(\nu)] \, / \, \nabla U . \underline{c}_t(\nu) \qquad (t = 1,T).$$

---

[9] The strict concavity assumption for U ensures that the utility possibilities frontier obtained as we vary the weights ε_t is convex.

If after ν iterations we have

$$\varepsilon_t(\nu+1) = \varepsilon_t(\nu) \qquad (t = 1,T),$$

then the solution to the ν^{th} programming problem also satisfies the additional condition (4.11) and hence it represents a multiperiod equilibrium. Exogenous values of the weights are still required for the first iteration, but any values of the right order will suffice.

Intuitively, the sense of the iterative procedure can be ascertained from the relation :

$$\varepsilon_t(\nu+1) - \varepsilon_t(\nu) =$$
$$\{[1-s][w_t(\nu)L_t + \underline{\Pi}_t'(\nu)\underline{k}_t(\nu)] - \underline{\phi}_t'(\nu)\underline{c}_t(\nu)\} / \nabla U.\underline{c}_t(\nu)$$

implied by (4.9), (4.12) and (4.13). When the share of income

$$[1-s][w_t(\nu)L_t + \underline{\Pi}_t'(\nu)\underline{k}_t(\nu)]$$

allocated to consumption in period t exceeds the cost

$$\underline{\phi}_t'(\nu)\underline{c}_t(\nu)$$

of the commodities consumed, the discount factor ε_t for that period is increased in the next iteration; that is, the rate of discount $(1-\varepsilon_t)$ is reduced. This will tend to induce a higher level of consumption for period t in the next iteration and reduce the excess of consumption funds. Equilibrium is achieved when the excess is reduced to zero.

Dixon (1976) has found that this kind of algorithm converges rapidly in practice. His models involved the joint maximization of utility across various consumer groups rather than across time periods, but the correspondence is close in computational terms, and we can expect similar convergence properties.

5. CONCLUSION

The Kuhn-Tucker conditions for a solution to a multisector programming model usually include at least some which are suggestive of market equilibrium. In this paper, we have examined the extent of this correspondence for a representative multiperiod model in elementary form. The Kuhn-Tucker conditions do define an equilibrium but one in which consumers require an implausible amount of information. This requirement is relaxed when a savings constraint is included in the model; then no economic agent needs information on prices and incomes more than one period ahead for equilibrium to be accomplished.

We conclude that multiperiod programming models correspond closely with competitive general equilibrium, at least in their elementary form. Although aspects of the general equilibrium interpretation of a particular model may be strained, it is possible to formulate sensible programming problems which undoubtedly describe general equilibrium.

The programming problem for the modified version of the model is largely unchanged from the original version, the substantive difference being that the weights for summing consumption over time, which were previously exogenous, must now satisfy an endogenous side relation. The side relation was obtained by first adjusting the equilibrium conditions to include the savings constraint and then deriving the corresponding programming problem. The sequence here is important for, without the benefit of an a priori formulation of the equilibrium conditions, the form required for the side relation is most obscure.

Hence we may also conclude that the general equilibrium approach offers greater insight into the nature of a multisector planning model and facilitates model formulation.

REFERENCES

1. Dixon, Peter B., The Theory of Joint Maximization, (Amsterdam : North Holland, 1975).

2. Dixon, Peter B., "The Computation of Economic Equilibria : A Joint Maximization Approach", Seminar Paper No. 47, Department of Economics, Monash University, 1976.

3. Kendrick, David, "Mathematical Models in Economic Planning," mimeographed, University of Texas, 1970. To be published by Holden Day.

4. Kendrick, David, "Systems Problems in Economic Development," in J.M. English (ed.), Economics of Engineering and Social Systems, (New York : Wiley-Interscience, 1972).

5. Mangasarian, Olvi L., Nonlinear Programming, (New York : McGraw Hill, 1969).

6. Taylor, Lance, "Theoretical Foundations and Technical Implications", in Charles R. Blitzer, Peter B. Clark and Lance Taylor (eds.), Economy-Wide Models and Development Planning, (London : Oxford University Press, 1975).