IMPACT OF DEMOGRAPHIC CHANGE ON INDUSTRY STRUCTURE IN AUSTRALIA

A joint study by the Australian Bureau of Statistics, the Industries Assistance Commission, the Department of Labor and Immigration and the Department of Manufacturing Industry.

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A JOINTMAX ALGORITHM FOR THE SOLUTION OF SNAPSHOT

by

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1. INTRODUCTION

The SNAPSHOT model was described in another paper.[1] For convenience I have reproduced both table 1 from that paper (with some minor alterations) and the appendix listing the variables. The aim of this paper is to suggest a method for computing solutions for the SNAPSHOT model, i.e., a method for solving the system of equations and inequalities in table 1.

In applications of the SNAPSHOT model, we will have a data base with up to 100 commodities (i.e., $n = 100$), and perhaps 10 consumer groups (i.e., $m = 10$). It quickly becomes apparent, on inspection of table 1, that SNAPSHOT involves a very large and non-linear system of equations and inequalities. Any direct solution method which failed to take advantage of special features arising from the economic nature of the problem, would be prohibitively expensive both in computer costs and programming effort. A potentially feasible approach is provided by the method of joint maximization.

The central theoretical idea of joint maximization can be traced back to Negishi [1960]. He showed that solutions for a particular class of general equilibrium models (those in which consumers are assumed

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[1] See Dixon, Harrower and Powell [1976].

to be constrained utility maximizers) can be generated by solving a "suitably" chosen mathematical programming problem. Iterative procedures for finding the appropriate programming problem have been suggested and applied by Dixon [1975a, 1975b], Osterrieth and Waelbroeck [1975], and Ginsburgh and Waelbroeck [1974]. The present paper modifies this earlier work to make it applicable in the SNAPSHOT model.

It may be useful at the outset to mention some issues on which I will have nothing to contribute. First, the existence question. I will be simply assuming that a SNAPSHOT solution exists - if this hypothesis is false I expect that we will find out via the computations. Second, I will not be proving any convergence propositions, but I will be returning to this issue in the conclusion. Third, I cannot demonstrate uniqueness, i.e., I cannot guarantee that there is no more than one SNAPSHOT solution. (Dixon, Harrower and Powell [1976] does the usual but inadequate equation and variable counting exercise.) To me, the uniqueness question is the most disturbing. However, at this stage, it appears that (along with most applied economists) we will have to be content with recomputing from different starting points. A second partial test for uniqueness, which could also be applied, is discussed by Dixon and Butlin [1975, appendix 2].

Rather than following the usual practice of concluding the introduction with a listing of the structure of the paper, I have included a flow diagram. There are several lengthy strings of equations in the paper, and hopefully the reader can refer back to figure 1 if he loses sight of where I am trying to go.

TABLE 1 : THE EQUATIONS SPECIFYING THE SNAPSHOT MODEL

Equation No.	Equation	No. of Equation Equivalents	Description
(1.1)	$C_i = f_i(p, Z_i),\ i=1,\ldots,m$	mn	Consumer demand functions
(1.2)	$Z_i = (1 - s_i)\alpha_i(GNP)$	m	Level of total private expenditure
(1.3)	$K(t) = (I + \hat{h})^t(\overline{K(0)})$	n	Capital stocks in snapshot year
(1.4)	$K(t + 1) = (I + \hat{h})(K(t))$	n	Post-snapshot year capital stocks
(1.5)	$J = K(t + 1) - (I - \hat{\eta})(K(t))$	n	Gross investments
(1.6)(a) (1.6)(b)	$X \leq K(t)$, and $\hat{\Pi}\ (X - K(t)) = 0$	n	Capacity constraint and complementary slack condition
(1.7)	$r = \beta\bar{r}$	n	Absolute rate of return on capital
(1.8)(a) (1.8)(b)	$r \geq \widehat{(p'K)}^{-1}\ \Pi - \eta$ $\hat{J}\ (r - \widehat{(p'K)}^{-1}\ \Pi + \eta) = 0$	n	Rate of return on capital and complementary slack condition
(1.9)	$E = \bar{E}$	n	Level of exports
(1.10)(a) (1.10)(b)	$M \leq \hat{\gamma}X$, and $\hat{\phi}\ (M - \hat{\gamma}X) = 0$	n	Import restraint and complementary slack condition
(1.11)	$p = \Theta\overline{p^e} + \xi$	n	Export price equation
(1.12)(a) (1.12)(b)	$p' \leq \Theta(\overline{p^m})'(I + \hat{\bar{\tau}}) + \phi'$, and $[p' - \Theta(\overline{p^m})'(I + \hat{\bar{\tau}}) - \phi']\hat{M} = 0$	n	Import price equation and complementary slack condition

... *continued*

Table 1 continued ....

Equation No.	Equation	No. of Equation Equivalents	Description
(1.13)(a)	$\bar{B} \geq (\overline{p^m})'M - (\overline{p^e})'E$	1	Balance of trade
(1.13)(b)	$\Theta(\bar{B} - (\bar{p}^m)'M + (\bar{p}^e)'E) = 0$		
(1.14)(a)	$p'(I - A) - w'\ell - \Pi' \leq 0$	n	Commodity cost structure and complementary slack condition
(1.14)(b)	$[p'(I - A) - w'\ell - \Pi']\hat{X} = 0$		
(1.15)(a)	$X + M \geq \sum_{i=1}^{m} C_i + KJ + \bar{G} + E + AX$	n	Product market clearing and complementary slack condition
(1.15)(b)	$\hat{p}\,[X + M - \sum_{i=1}^{m} C_i - KJ - \bar{G} - E - AX] = 0$		
(1.16)(a)	$\bar{N} \geq 1'L$	1	Full employment of total labour force
(1.16)(b)	$\delta(\bar{N} - 1'L) = 0$		
(1.17)	$L = \ell X$	H	Production labour requirements
(1.18)	$w = \delta(\bar{w})$	H	Sets wage relativities
(1.19)	$GNP = w'L + \Pi'K(t) + [\Theta(\overline{p^m})'\hat{\bar{\tau}} + \phi']M - \xi'E$	1	Gross national product
(1.20)	All endogenous variables (with the possible exception of h and ξ) must be non-negative		Sign constraints

DEFINITION OF NOTATION

Endogenous Variables in the Snapshot Year

		Number of Variables
C_i	consumption of commodities by consumer group i	(nm)
p	commodity prices	(n)
Z_i	total expenditure of consumer group i	(m)
GNP	gross national product	(1)
h	average rate of growth of capital in each industry over the t-year snapshot period	(n)
$K(t)$	industry levels of capital stock in the snapshot year	(n)
$K(t+1)$	industry levels of capital stock in the year after the snapshot year	(n)
J	gross investments by using industries	(n)
X	outputs of commodities	(n)
Π	rental prices on capital by industries	(n)
r	minimum acceptable rates of return by industry	(n)
β	variable reflecting the absolute rate of return demanded on new capital formation for Australian industry	(1)
E	exports of commodities (quantity)	(n)
M	imports of commodities (quantity)	(n)
Θ	exchange rate ($A per unit of foreign currency)	(1)

continued ...

		Number of Variables
ϕ	excess tariff revenue per unit of imports	(n)
ξ	export tax (ξ_j positive) or subsidy (ξ_j negative)	(n)
w	wage rates by occupation before taxes	(H)
L	the number of labor units in each occupational group in the snapshot year	(H)
δ	variable reflecting the absolute level of wages before taxes for the Australian labor force	(1)

Exogenous Variables in the Snapshot Year

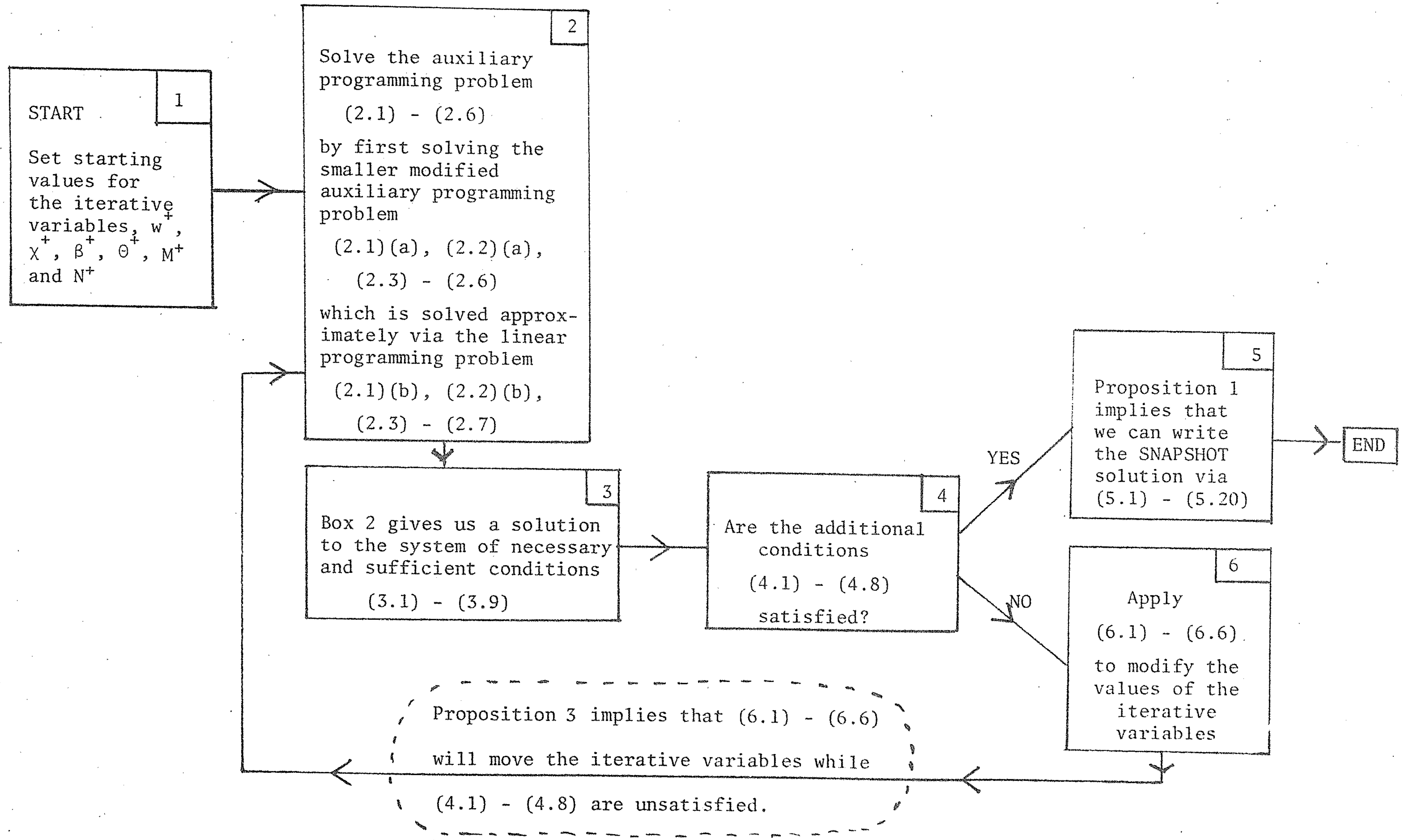
		Number of Variables
s_i	consumer group i's average propensity to save out of disposable income	(m)
α_i	share of GNP which is disposable income for group i	(m)
$\overline{K(0)}$	industry levels of capital stock in the base year	(n)
t	number of years of the snapshot period	(1)
η	industry specific depreciation rates applicable to the the industry capital stocks, K(t), over the tth year	(n)
K	capital matrix in the snapshot year, K_{ij} is the input of good i required to create a unit of capital stock for industry j	(n × n)
$\overline{r}$	relative rates of return to capital required to induce investment in each industry	(n)
$\overline{E}$	exports of commodities	(n)
γ	import shares of the domestic markets	(n)
$\overline{p^e}$	export prices (f.o.b.) in foreign currency	(n)
$\overline{p^m}$	import prices (c.i.f.) in foreign currency	(n)

continued ...

		Number of Variables
$\bar{\tau}$	ad valorem tariff rates	(n)
$\bar{B}$	balance of trade deficit in foreign currency	(1)
A	input-output coefficients matrix	(n × n)
ℓ	labor requirements by occupation and industry per unit of output in the snapshot year	(H × n)
$\bar{G}$	government purchases of commodities	(n)
$\bar{N}$	total number of people in the workforce in the snapshot year	(1)
$\bar{w}$	relative wage rates, before taxes, for the various occupational groups	(H)

In addition to the above list of exogenous variables, U_i, the utility function for the i^{th} consumer group, will be exogenously specified.

Figure 1 : Flow Chart for Jointmax Solution of SNAPSHOT



2. THE AUXILIARY PROGRAMMING PROBLEM

Given values for the set of data matrices, scalars and vectors

$$D = \{s_1 \ldots s_m, \alpha_1 \ldots \alpha_m, \overline{K(0)}, t, \eta, K, \bar{r}, \bar{E}, \gamma, \bar{p}^e, \bar{p}^m, \bar{\tau},$$
$$\bar{B}, A, \ell, \bar{G}, \bar{N}, \bar{w}\} ,$$

our problem is to compute values for the set of endogenous variables

$$\Xi(D, \psi) = \{C_1 \ldots C_m, p, Z_1 \ldots Z_m, GNP, h, K(t), K(t+1), J,$$
$$X, \Pi, r, \beta, E, M, \Theta, \phi, \xi, w, L, \delta\}$$

to satisfy the system ψ, where ψ is the list of equations and inequalities in table 1, i.e., given D, find $\Xi(D, \psi)$ to satisfy ψ.

As a first step, we consider the following programming problem:

Choose X_1, X_2, $C_1 \ldots C_m$, M, all non-negative n-order vectors, to maximize

$$\sum_i w_i^+ U_i(C_i) - \Theta^+ (\bar{p}^m)' (\hat{\bar{\tau}}) M , \tag{2.1}$$

subject to

$$- (I - A)(X_1 + X_2) + K(\hat{\bar{r}}\beta^+ + \hat{\eta})X_2 + \sum_i C_i + \bar{G} + \bar{E} - M + \chi^+ \leqslant 0 , \tag{2.2}$$

$$M - M^+ \leqslant 0 \tag{2.3}$$

$$\bar{w}' \ell (X_1 + X_2) - N^+ \leqslant 0 , \tag{2.4}$$

$$(2.5) \qquad X_1 - (I - \hat{\eta})^t K(0) \leqslant 0 \quad ,$$

and

$$(2.6) \qquad (\bar{p}^m)'M - (\bar{p}^e)'\bar{E} - \bar{B} \leqslant 0 \quad ,$$

where U_i is a strictly concave[1] utility function describing the preferences of the i^{th} consumer group. (The demand functions (1.1) are obtained by choosing C_i to maximize $U_i(C_i)$ subject to $p'C_i = Z_i$). The notation $(\hat{\ })$ denotes the diagonal matrix formed from the elements of the vector ( ), and the superscript + denotes <u>iterative variable</u>. The iterative variables are the scalars $w_1^+, w_2^+, \ldots w_m^+, \beta^+, \Theta^+$ and N^+, and the n-order vectors χ^+ and M^+. All the iterative variables, with the exception of χ^+ are restricted to non-negative values, and each is treated as a parameter in the problem (2.1) - (2.6). In computing Ξ, we will solve a sequence of problems of the form (2.1) - (2.6), where the values of the iterative variables will be changed systematically from problem to problem.

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[1] The use of strictly concave utility functions is convenient for computations. It ensures that the problem (2.1) - (2.6) has no more than one solution. From the viewpoint of the economics, strict concavity is not a restrictive assumption. In fact, it is no more restrictive than the assumption that consumers are utility maximizers, see Dixon [1975a, pp. 96-105]. We will also assume that U_i exhibits non-satiation, i.e., for any consumption vector c, and any neighbourhood, N(c) of c, there exists $\tilde{c} \varepsilon N(c)$ with $\tilde{c} \neq c$, $\tilde{c} \geqslant c$ and $U_i(\tilde{c}) > U_i(c)$.

Before describing the iterative procedure, it is convenient to set out the necessary and sufficient conditions for a solution of (2.1) - (2.6).[1] X_1, X_2, C_1, ... C_m, M, all non-negative, is a solution for (2.1) - (2.6) if and only if there exist non-negative scalars δ, Θ and non-negative n-order vectors p, ϕ, and Π, such that X_1, X_2, C_1, ... C_m, M, δ, Θ, p, ϕ and Π jointly satisfy the system

(3.1)(a) $$w_i^+\nabla U_i(C_i) \leq p \quad , \qquad i = 1 \ldots m$$

(b) $$\hat{C}_i(w_i^+\nabla U_i(C_i) - p) = 0 \quad , \qquad i = 1 \ldots m \; ;$$

(3.2)(a) $$p'(I - A) - \delta\bar{w}'\ell - \Pi' \leq 0$$

(b) $$(p'(I - A) - \delta\bar{w}'\ell - \Pi')\hat{X}_1 = 0 \; ;$$

(3.3)(a) $$p'(I - A) - \delta\bar{w}'\ell - p'K(\hat{\bar{r}}\beta^+ + \hat{\eta}) \leq 0$$

(b) $$[p'(I - A) - \delta\bar{w}'\ell - p'K(\hat{\bar{r}}\beta^+ + \hat{\eta})]\,\hat{X}_2 = 0 \; ;$$

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[1] I assume that a constraint qualification is satisfied (e.g., Slater's condition that $\exists\, X_1, X_2, C_1, \ldots, C_m, M > 0$ satisfying the constraints as strict inequalities.) Then the strict concavity of the U_i ensures that (3.1) - (3.9) are both necessary and sufficient conditions.

(3.4)(a) $$p' - \Theta^{+}(\bar{p}^{m})'\hat{\bar{\tau}} - \Theta(\bar{p}^{m})' - \phi' \leqslant 0$$

(b) $$(p' - \Theta^{+}(\bar{p}^{m})'\hat{\bar{\tau}} - \Theta(\bar{p}^{m})' - \phi')\hat{M} = 0 \quad ;$$

(3.5)(a) $$-\ (I - A)(X_1 + X_2) + K(\hat{\bar{r}}\beta^{+} + \hat{\eta})X_2 + \sum_i C_i + \bar{G} + \bar{E} - M + \chi^{+} \leqslant 0$$

(b) $$\hat{p}[-\ (I - A)(X_1 + X_2) + K(\hat{\bar{r}}\beta^{+} + \hat{\eta})X_2 + \sum_i C_i + \bar{G} + \bar{E} - M + \chi^{+}] = 0 \quad ;$$

(3.6)(a) $$M - M^{+} \leqslant 0$$

(b) $$\hat{\phi}\ [M - M^{+}] = 0 \quad ;$$

(3.7)(a) $$\bar{w}'\ell(X_1 + X_2) - N^{+} \leqslant 0$$

(b) $$\delta(\bar{w}'\ell(X_1 + X_2) - N^{+}) = 0 \quad ;$$

(3.8)(a) $$X_1 - (I - \hat{\eta})^{t}\ \overline{K(0)} \leqslant 0$$

(b) $$\hat{\Pi}(X_1 - (I - \hat{\eta})^{t}\ \overline{K(0)}) = 0 \quad ;$$

(3.9)(a) $$(\bar{p}^{m})'M - (\bar{p}^{e})'\bar{E} - \bar{B} \leqslant 0$$

(b) $$\Theta((\bar{p}^{m})'M - (\bar{p}^{e})'\bar{E} - \bar{B}) = 0 \quad .$$

The relevance of problem (2.1) - (2.6) in computing a SNAPSHOT solution $\Xi(D,\psi)$ can now be made apparent.

Proposition 1 :

Assume that

$$S^* = \{X_1^*, X_2^*, C_1^*, \ldots, C_m^*, M^*, \delta^*, \Theta^*, p^*, \phi^*, \Pi^*\}$$

is a solution to the system (3.1) - (3.9). (S^* can be computed by solving the problem (2.1) - (2.6) and its dual.) In addition, assume that

$$(4.1) \qquad (p^*)'C_i^* = (1 - s_i)\alpha_i \, GNP^* , \qquad i = 1 \ldots m ,$$

where

$$(4.2) \qquad GNP^* = \delta^* \bar{w}' \ell(X_1^* + X_2^*) + (\Pi^*)'((I - \hat{\eta})^t \, \overline{K(0)} + X_2^*)$$
$$+ [\Theta^+ (\bar{p}^m)' \hat{\bar{\tau}} + (\phi^*)']M^* - [(p^*)' - \Theta^* (\bar{p}^e)']\bar{E} ,$$

$$(4.3) \qquad \hat{\gamma}(X_1^* + X_2^*) = M^+ ,$$

$$(4.4)(a) \qquad 1'\ell(X_1^* + X_2^*) \leq \bar{N}$$

$$(b) \qquad \delta^*(1'\ell(X_1^* + X_2^*) - \bar{N}) = 0 ,$$

where $1'$ is the n-order row vector of ones,

$$(4.5) \qquad \Theta^* = \Theta^+ ,$$

(4.6) $$\chi^{+} = - K(\hat{\bar{r}}\beta^{+} - \hat{h}^{*})X_2^{*} + K(\hat{h}^{*} + \hat{\eta})(I - \hat{\eta})^{t}\ \overline{K(0)} \quad ,$$

where h^{*} is an n × 1 vector with typical element

(4.7) $$h_j^{*} = \left(\frac{(1 - \eta_j)^{t}\,\overline{K_j(0)} + X_{2j}^{*}}{\bar{K}_j(0)} \right)^{1/t} - 1 \quad ,$$

and

(4.8) $$w_1^{+}, \ldots, w_m^{+}, M^{+}, \beta^{+}, \theta^{+}, N^{+} \text{ and } M^{+} \geqslant 0 \quad .$$

Then Ξ^{*}, defined by

(5.1) $$C_i = C_i^{*} \quad , \qquad i = 1 \ldots m \quad ,$$

(5.2) $$p = p^{*} \quad .$$

(5.3) $$Z_i = (p^{*})'C_i^{*} \quad , \qquad i = 1 \ldots m \quad ,$$

(5.4) $$GNP = GNP^{*} \quad ,$$

(5.5) $$h = h^{*} \quad ,$$

(5.6) $$K(t) = (I - \hat{\eta})^{t}\ \overline{K(0)} + X_2^{*} \quad ,$$

(5.7) $$K(t + 1) = (I + \hat{h}^{*}) \{(I - \hat{\eta})^{t}\ \overline{K(0)} + X_2^{*}\} \quad ,$$

$$J = (I + \hat{h}^*)\{(I - \hat{\eta})^t\,\overline{K(0)} + X_2^*\} - (I - \hat{\eta})^{t+1}\,\overline{K(0)} - (I - \hat{\eta})X_2^* , \tag{5.8}$$

$$X = X_1^* + X_2^* , \tag{5.9}$$

$$\Pi = \Pi^* \tag{5.10}$$

$$r = \bar{r}\beta^+ , \tag{5.11}$$

$$\beta = \beta^+ , \tag{5.12}$$

$$E = \bar{E} , \tag{5.13}$$

$$M = M^* , \tag{5.14}$$

$$\Theta = \Theta^* , \tag{5.15}$$

$$\phi = \phi^* , \tag{5.16}$$

$$\xi = p^* - \Theta^*(\bar{p}^e) , \tag{5.17}$$

$$w = \delta^*\bar{w} , \tag{5.18}$$

$$L = \ell(X_1^* + X_2^*) , \tag{5.19}$$

$$\delta = \delta^* \tag{5.20}$$

is a solution to the SNAPSHOT problem, i.e., (5.1) - (5.20) is a solution to the system in table 1.

Proof:

(a) From (3.1) we may conclude that

$$C_i^* = f_i(P^*, (p^*)' C_i^*) \quad .$$

(Since U_i is strictly concave and exhibits non-satiation, (3.1) is both necessary and sufficient to ensure that C_i^* maximizes $U_i(C_i)$, subject to $(p)' C_i = (p^*)' C_i^*$ and $C_i \geqslant 0$.)

Then by referring to (5.1) - (5.3), we see that Ξ^* satisfies (1.1).

(b) (4.1) implies, via (5.3) and (5.4) that Ξ^* satisfies (1.2).

(c) (4.7) implies that

$$\bar{K}_j(0)(1 + h_j^*)^t = (1 - \eta_j)^t \bar{K}_j(0) + X_{2j}^* \quad ;$$

then (5.6) and (5.5) imply that Ξ^* satisfies (1.3).

(d) (5.5) - (5.7) imply that Ξ^* satisfies (1.4).

(e) (5.6) - (5.8) imply that Ξ^* satisfies (1.5).

(f) (3.8) implies that

$$X_1^* + X_2^* \leqslant (I - \hat{\eta})^t \bar{K}(0) + X_2^* \quad ,$$

and that

$$\hat{\Pi}^*(X_1^* + X_2^* - (I - \hat{\eta})^t \bar{K}(0) - X_2^*) = 0 \quad .$$

Referring to (5.9), (5.6) and (5.10), we see that Ξ^* satisfies (1.6).

(g) (5.11) and (5.12) imply that Ξ^* satisfies (1.7).

(h) Assume that

$$\text{(h1)} \qquad \beta^+\bar{r}_j < [(\widehat{(p^*)'K})^{-1}\Pi^*]_j - \eta_j$$

where $[\ ]_j$ denotes the j^{th} element of the vector $[\ ]$.

Then

$$[(p^*)'(I - A) - \delta^*\bar{w}'\ell - (\Pi^*)']_j$$

$$< [(p^*)'(I - A) - \delta^*\bar{w}'\ell - (p^*)'K(\hat{\bar{r}}\beta^+ + \hat{\eta})]_j$$

and it follows from (3.2) and (3.3) that $X^*_{1j} = 0$. (3.8)(b) now implies that $\Pi^*_j = 0$. Hence (h1) implies that $\beta^+ < 0$. This contradicts (4.8), and it follows that

$$\text{(h2)} \qquad \beta^+\bar{r} \geq (\widehat{(p^*)'K})^{-1}\Pi^* - \eta .$$

Next, we assume that

$$\text{(h3)} \qquad \beta^+\bar{r}_j > [(\widehat{(p^*)'K})^{-1}\Pi^*]_j - \eta_j .$$

Following a similar argument to the one just given we can establish that (h3) implies that $X^*_{2j} = 0$. Hence from (4.7) we have $h^*_j = -\eta_j$.

We may conclude that

$$\text{(h4)}\qquad \left[\beta + \bar{r}_j - [\widehat{((p^*)'K)}^{-1}\,\Pi^*]_j + \eta_j\right]\Big[(1 + h_j^*)[(1 - \eta_j)^t\,\bar{K}_j(0) + X_{2j}^*] - (1 - \eta_j)^{t+1}\,\bar{K}_j(0) - (1 - \eta_j)X_{2j}^*\Big] = 0 .$$

Referring to (5.2), (5.10), (5.11) and (5.8) we see that (h2) and (h4) ensure that Ξ^* satisfies (1.8)(a) and (b).

(i) (5.13) implies that Ξ^* satisfies (1.9).

(j) From (3.6)(a) and (b) and (4.3)

$$M^* \leqslant \hat{\gamma}(X_1^* + X_2^*)$$

$$\hat{\phi}^*\,[M^* - \hat{\gamma}(X_1^* + X_2^*)] = 0 .$$

Then from (5.14), (5.9) and (5.16) it follows that Ξ^* satisfies (1.10).

(k) Referring to (5.17), (5.2) and (5.15) we see that Ξ^* satisfies (1.11).

(l) From (3.4)(a) & (b), and (4.5) we have

$$(p^*)' - \Theta^*(\bar{p}^m)'\hat{\bar{\tau}} - \Theta^*(\bar{p}^m)' - (\phi^*)' \leqslant 0$$

and

$$((p^*)' - \Theta^*(\bar{p}^m)'\hat{\bar{\tau}} - \Theta^*(\bar{p}^m)' - (\phi^*)')\,\hat{M}^* = 0 .$$

Then referring to (5.2), (5.15), (5.16) and (5.14) we see that Ξ^* satisfies (1.12)(a) & (b).

(m) (3.9)(a) & (b) imply that

$$(\bar{p}^m)'M^* - (\bar{p}^e)'\bar{E} - \bar{B} \leqslant 0$$

and

$$\Theta^* \left((\bar{p}^m)'M^* - (\bar{p}^e)'\bar{E} - \bar{B} \right) = 0 .$$

Then (5.14), (5.13) and (5.15) imply that Ξ^* satisfies (1.13).

(n) From (3.2)(a) & (b) we can write

$$\text{(n1)} \qquad (p^*)'(I - A) - \delta^* \bar{w}\ell - (\Pi^*)' \leqslant 0$$

and

$$\text{(n2)} \qquad \left((p^*)'(I - A) - \delta^* \bar{w}'\ell - (\Pi^*)' \right) \hat{X}_1^* = 0 .$$

If $[(p^*)'(I - A) - \delta^* \bar{w}'\ell - (\Pi^*)']_j < 0$,

then it follows from (h2) that

$$[(p^*)'(I - A) - \delta^* \bar{w}'\ell - (p^*)'K(\beta + \hat{\bar{r}} + \hat{\eta})]_j < 0 ,$$

and hence, from (3.3)(b), $X_{2j}^* = 0$. Therefore, we can rewrite (n2) as

$$\text{(n3)} \qquad \left((p^*)'(I - A) - \delta^* \bar{w}\ell - (\Pi^*)' \right)(\hat{X}_1^* + \hat{X}_2^*) = 0 .$$

(n1) and (n3), in conjunction with (5.2), (5.18), (5.10) and (5.9), imply that Ξ^* satisfies (1.14)(a) & (b).

(o) (3.5)(a) & (b) and (4.6) imply that

$$- (I - A)(X_1^* + X_2^*) + \sum_i C_i^* + \bar{G} + \bar{E} - M^* +$$
$$K(\hat{h}^* + \hat{\eta})((I - \hat{\eta})^t \, \overline{K(0)} + X_2^*) \leqslant 0 ,$$

and

$$\hat{p}^*[-(I - A)(X_1^* + X_2^*) + \sum_i C_i^* + \bar{G} + \bar{E} - M^* +$$
$$K(\hat{h}^* + \hat{\eta})((I - \hat{\eta})^t \bar{K}(0) + X_2^*)] = 0 .$$

Then referring to (5.9), (5.1), (5.13), (5.14), (5.8) and (5.2) we see that Ξ^* satisfies (1.15)(a) & (b).

(p) From (4.4), (5.19) and (5.20), it follows that Ξ^* satisfies (1.16).

(q) (5.19) and (5.9) imply that Ξ^* satisfies (1.17).

(r) (5.18) and (5.20) imply that Ξ^* satisfies (1.18).

(s) (5.4), (5.18), (5.19), (5.10), (5.6), (5.15), (5.16), (5.14), (5.17), (5.13), (4.5) and (4.2) together imply that Ξ^* satisfies (1.19).

(t) All the vectors and scalars in S^* are non-negative. Also, the iterative variables (see (4.8)) are non-negative. Finally, $h_j^* > -\eta_j$, see (4.7), and $\eta_j < 1$.

Referring to (5.1) - (5.20) and (4.1), it is easy to see that Ξ^* satisfies the sign restrictions (5.20).

This completes the proof of proposition 1.

Proposition 1 means that if we solve problem (2.1) - (2.6), thereby generating a solution for the system (3.1) - (3.9), and this solution meets the additional requirements (4.1) - (4.8), then we can deduce a SNAPSHOT solution. However, proposition 1, by itself, is not sufficient to justify the study of problem (2.1) - (2.6) as a possible vehicle for generating SNAPSHOT solutions.

Perhaps it is possible for there to be a SNAPSHOT solution, Ξ^*, which cannot be revealed by problem (2.1) - (2.6), i.e., perhaps it is possible that there are no values for the iterative variables such that the solution of (2.1) - (2.6) reveals Ξ^*. Proposition 2 allays this doubt.

<u>Proposition 2</u>:

Let

$$\Xi^* = \{C_1^*, \ldots, C_m^*, p^*, Z_1^*, \ldots, Z_m^*, GNP^*, h^*, \overset{*}{K}(t), \overset{*}{K}(t+1),$$
$$J^*, X^*, \Pi^*, r^*, \beta^*, E^*, M^*, \Theta^*, \phi^*, \xi^*, w^*, L^*, \delta^*\}$$

be a SNAPSHOT solution, i.e., Ξ^* is a solution to the system in table 1. Set the iterative variables as follows:

$$w_i^+ = 1/Q_i(p^*, Z_i^*), \quad i = 1 \ldots m ,$$

$$\beta^+ = \beta^* ,$$

$$\Theta^+ = \Theta^* ,$$

$$N^+ = \bar{w}'\ell X^* ,$$

$$\chi^+ = - K(\hat{\bar{r}}\beta^* - \hat{h}^*)(X^* - \min\{(I - \hat{\eta})^t K(0), X^*\})$$
$$+ K(\hat{h}^* + \hat{\eta})(I - \hat{\eta})^t \bar{K}(0) ,$$

and

$$M^+ = \hat{\gamma}X^* ,$$

where Q_i is the marginal utility of expenditure for consumer i and is a function of prices and expenditure level. ($Q_i(p, Z_i)$ is the Lagrangian multiplier in the problem of choosing C_i to maximize $U_i(C_i)$ subject to $p'C_i = Z_i$.)

Then $\mathcal{S}^*$ defined by

$$
\begin{aligned}
X_1 &= \min\{(I-\hat{\eta})^t\,\bar{K}(0),\ X^*\}\\
X_2 &= X^* - \min\{(I-\hat{\eta})^t\,\bar{K}(0),\ X^*\}\\
C_i &= C_i^*, \quad i = 1 \ldots m\\
M &= M^*\\
\delta &= \delta^*\\
\theta &= \theta^*\\
p &= p^*\\
\phi &= \phi^*\\
\Pi &= \Pi^*
\end{aligned}
$$

satisfies the system (3.1) - (3.9). Also, it satisfies the additional conditions (4.1) - (4.8) and reveals the SNAPSHOT solution Ξ^* via (5.1) - (5.20).

As with proposition 1, the proof is rather laborious, and will be omitted. To prove the proposition, the interested reader can substitute from $\mathcal{S}^*$ into each of the relations (3.1) - (3.9), and use (1.1) - (1.20) to show that none is violated.

In summary, propositions 1 and 2 mean that (a) any solution for the system (3.1) - (3.9) (which can be solved through the auxiliary programming problem (2.1) - (2.6)) satisfying the additional conditions (4.1) - (4.8), reveals a SNAPSHOT solution via (5.1) - (5.20), and (b) every SNAPSHOT solution can be generated via the problem (2.1) - (2.6), i.e., for any particular SNAPSHOT solution, there exist values for the iterative variables such that it may be generated by (2.1) - (2.6).

3. THE ALGORITHM

Our plan is to solve problem (2.1) - (2.6) with the iterative variables assigned arbitrary values $w_1^+(1)$, ..., $w_m^+(1)$, $\chi^+(1)$, $\beta^+(1)$, $\theta^+(1)$, $M^+(1)$ and $N^+(1)$. When we compute a solution to (2.1) - (2.6), and hence to (3.1) - (3.9), we cannot expect the additional conditions (4.1) - (4.8) to be satisfied. However, we can change the values of the iterative variables so that the v^{th} time, $v > 1$, we solve (2.1) - (2.6), the iterative variables are defined by

$$(6.1)\qquad M^{+}(v) = M^{+}(v-1) + \hat{\Lambda}_1(v)\left[\hat{\gamma}(X_1^{*}(v-1) + X_2^{*}(v-1)) - M^{+}(v-1)\right],$$

where the diagonal elements of $\hat{\Lambda}_1(v)$ are between 0 and 1, and $X_1^{*}(v-1)$, $X_2^{*}(v-1)$, etc., denote the values of X_1, X_2, etc. revealed by our $(v-1)^{th}$ solution of (2.1) - (2.6).

(6.2) If $\delta^{*}(v-1) = 0$ and $\bar{N} \geq 1'\ell(X_1^{*}(v-1) + X_2^{*}(v-1))$, then $N^{+}(v) = N^{+}(v-1)$.

Otherwise $N^{+}(v) = N^{+}(v-1)\bar{N}/1'\ell(X_1^{*}(v-1) + X_2^{*}(v-1))$.

$$(6.3)\qquad \chi^{+}(v) = -K(\hat{\bar{r}}\beta^{+}(v) - \hat{h}^{*}(v-1))\, X_2^{*}(v-1) + K(\hat{h}^{*}(v-1) + \hat{\eta})(I - \hat{\eta})^{t}\, \bar{K}(0) .$$

$$(6.4)\qquad \beta^{+}(v) = \beta^{+}(v-1) + \Lambda_2(v)\left[\sum_i \alpha_i(1 - s_i) - \frac{\sum_i (p^{*}(v-1))'C_i^{*}(v-1)}{GNP^{*}(v-1)}\right],$$

where $\Lambda_2(v)$ is a positive scalar chosen to ensure that $\beta^{+}(v) > 0$.

$$(6.5)\qquad \Theta^{+}(v) = \Theta^{*}(v-1)$$

and

$$(6.6) \qquad w_i^+(v) = \frac{1/Q_i(p^*(v-1),\ \alpha_i(1-s_i)GNP^*(v-1))}{\sum_{i=1}^{m}[1/Q_i(p^*(v-1),\ \alpha_i(1-s_i)GNP^*(v-1)]} \ ,$$

$$i = 1, \ldots, m \ ,$$

where, as before, Q_i is the marginal utility of expenditure for consumer i.

In proposition 3 we check that a steady state for the system (6.1) - (6.6) can only occur if the iterative variables are set so that (3.1) - (3.9) reveals a SNAPSHOT solution via (5.1) - (5.20).

Proposition 3:

Assume that

$$(7.1) \qquad M^+(v) = M^+(v-1) \ ,$$

$$(7.2) \qquad N^+(v) = N^+(v-1) \ ,$$

$$(7.3) \qquad \chi^+(v) = \chi^+(v-1) \ ,$$

$$(7.4) \qquad \beta^+(v) = \beta^+(v-1) \ ,$$

$$(7.5) \qquad \theta^+(v) = \theta^+(v-1) \ ,$$

and

(7.6) $$w_i^+(v) = w^+(v - 1) , \qquad i = 1, \ldots, m .$$

Then the $(v - 1)^{th}$ solution of (3.1) - (3.9) reveals a SNAPSHOT solution via (5.1) - (5.20).

Proof:

Under (7.1) - (7.6) we may assume that $S^*(v)$ is identical to $S^*(v - 1)$ where $S^*(v)$ is the v^{th} solution of (3.1) - (3.9). We check that (7.1) - (7.6) imply that the $S^*(v - 1)$ satisfies (4.1) and (4.3) - (4.6), where $GNP^*(v - 1)$ and $h_j^*(v - 1)$ are defined according to (4.2) and (4.7) respectively.

(a) (3.1) implies that

(a1) $$w_i^+(v - 1) = 1/Q_i(p^*(v - 1), (p^*(v - 1))'C_i^*(v - 1)) ,$$

while (6.6) implies that

(a2) $$w_i(v) = \frac{1/Q_i(p^*(v - 1), \alpha_i(1 - s_i)GNP^*(v - 1))}{\sum_{i=1}^{m} [1/Q_i(p^*(v - 1), \alpha_i(1 - s_i)GNP^*(v - 1))]} .$$

If $\sum_{i=1}^{m} 1/Q_i(p^*(v - 1), \alpha_i(1 - s_i)GNP^*(v - 1)) \equiv \sum_i 1/Q_i(v - 1) < 1$, then the strict concavity (implying diminishing marginal utility of expenditure) of the U_i, and (7.6) mean that

(a3) $$\alpha_i(1 - s_i)GNP^*(v - 1) < (p^*(v - 1))'C_i(v - 1)$$

for all i.

Similarly, if $\sum_i 1/Q_i(v - 1) > 1$, then

$$\text{(a4)} \quad \alpha_i(1 - s_i)GNP^*(v - 1) > (p^*(v - 1))'C_i(v - 1) \quad \text{for all } i \text{ .}$$

Both (a3) and (a4) are incompatible with (7.4), (see (6.4)). Hence, $\Sigma_i 1/Q_i(v - 1) = 1$.

Then (a1), (a2) and (7.6) imply that

$$Q_i(p^*(v - 1), (p^*(v - 1))'C_i^*(v - 1))$$

$$= Q_i(p^*(v - 1), \alpha_i(1 - s_i)GNP^*(v - 1)) \text{ .}$$

Hence, $(p^*(v - 1))'C_i^*(v - 1) = \alpha_i(1 - s_i)GNP^*(v - 1)$ and $S^*(v - 1)$ satisfies (4.1)

(b) (6.1) and (7.1) imply that $S^*(v - 1)$ satisfies (4.3).

(c) Since $N^+(v) = N^+(v - 1)$, (6.2) implies that either $\delta^*(v - 1) = 0$ and $\bar{N} \geqslant 1'\ell(X_1^*(v - 1) + X_2^*(v - 1))$, or $\bar{N} = 1'\ell(X_1^*(v - 1) + X_2^*(v - 1))$. In either case, $S^*(v - 1)$ satisfies (4.4).

(d) (6.5) implies that $\Theta^+(v) = \Theta^*(v - 1)$. Also, $\Theta^+(v - 1) = \Theta^+(v)$, (see (7.5)) . Hence, $S^*(v - 1)$ satisfies (4.5).

(e) (6.3), (7.4) and (7.3) imply that

$$\chi^{+}(v - 1) = - K(\hat{\bar{r}}\beta^{+}(v - 1) - \hat{h}^{*}(v - 1))X_2^{*}(v - 1)$$

$$+ K(\hat{h}^{*}(v - 1) + \hat{\eta})(I - \hat{\eta})^{t}\,\overline{K}(0) .$$

Hence, $S^{*}(v - 1)$ satisfies (4.6).

This completes the proof of proposition 3.

The rationale for each of the adjustment rules (6.1) - (6.6) is quite straightforward.

M^{+} is our guess of the maximum allowable import levels (see (1.10)(a)), and (6.1) updates our guess on the basis of our latest estimate, $(X_1^{*}(v - 1) + X_2^{*}(v - 1))$, of the SNAPSHOT solution output vector. It may not be obvious to the reader as to why we used the iterative variable M^{+}. Can we replace (2.3) with the constraint

$$(8.1) \qquad M - \hat{\gamma}(X_1 + X_2) \leqslant 0 \text{ ?}$$

The problem with (8.1) is that it introduces an unwanted distortion into the cost inequalities (3.2) and (3.3). For example, under (8.1) (3.2)(a) would become

$$p'(I - A) - \delta\bar{w}\ell - \Pi + \phi'\hat{\gamma} \leqslant 0 .$$

At this stage we cannot say much about the choice of $\hat{\Lambda}_1(v)$. Computational experience will indicate whether cycling is a serious problem, e.g., when

$$\gamma_j(X^*_{1j}(v - 1) + X^*_{2j}(v - 1)) > M^+_j(v - 1)$$

(6.1) indicates that we should raise M^+_j, i.e.,

$$M^+_j(v) > M^+_j(v - 1) \quad .$$

When we increase M^+_j, it is likely that domestic output will fall, i.e.,

$$X^*_{1j}(v) + X^*_{2j}(v) < X^*_{1j}(v - 1) + X^*_{2j}(v - 1)$$

and it is possible that

$$\gamma_j(X^*_{1j}(v) + X^*_{2j}(v)) < M^+_j(v) \quad .$$

By choosing small values for $\Lambda_{1j}(v)$, $v > 1$, we can, if necessary, dampen the amplitude of the cycles.

N^+ is our guess of the number of "wage units" available in the snapshot year. The total number of people in the workforce, $\bar{N}$, is data. However, if we used the constraint

$$(8.2) \qquad 1'\ell(X_1 + X_2) - \bar{N} \leqslant 0$$

in place of (2.4), our necessary and sufficient conditions (3.1) - (3.9) would fail to reflect the wage differentials required by (1.18). In fact, (8.2) would impose a uniform wage across all occupations.

The idea behind (6.2) is as follows: if solution (v - 1) of (3.1) - (3.9) implies 5% greater employment than is available, i.e.,

$$1'\ell(X^*_1(v - 1) + X^*_2(v - 1)) = 1.05\ \bar{N} \quad ,$$

then we lower the availability of "wage units" by 5%, i.e.,

$N^+(v) = N^+(v - 1)/1.05$, which can be expected to lower employment by approximately 5%, i.e.,

$$\frac{1'\ell(X_1^*(v-1) + X_2^*(v-1))}{1'\ell(X_1^*(v) + X_2^*(v))} \simeq 1.05 \quad .$$

The role of the iterative variable χ^+ is to modify constraint (2.2) so that it correctly reflects the commodity balance equations (see (1.15(a))). In constraint (2.2) it was necessary to include the term $K(\hat{\bar{r}}\beta^+)X_2$. While this term has no place in a demand equals supply equation, it forces problem (2.1) - (2.6) to correctly reflect the rates of return defined by (1.7) - (1.8), (see part (h) in the proof of proposition 1). χ^+ attempts to eliminate the unwanted term, and replace it with the appropriate investment vector KJ.

The adjustment rule (6.3) updates χ^+ according to the most recent solution of problem (2.1) - (2.6) and the currently planned value for β^+. A similar adjustment rule was used by Dixon and Butlin [1975], and no difficulty was experienced.

The iterative variable β^+ is introduced to induce problem (2.1) - (2.6) to generate the appropriate level of savings, i.e., to force the restriction

$$p'\sum_i C_i = \text{GNP} \sum_i \alpha_i(1 - s_i) \ , \qquad \text{(see (1.2))} \ .$$

If problem (2.1) - (2.6) implies too much savings, i.e.,

$$\sum_i \alpha_i(1 - s_i) > \left[\sum_i (p^*(v-1))'C_i(v-1)\right] \Big/ \text{GNP}^*(v-1) \ ,$$

then (6.4) indicates that we should increase β^+ , i.e.,

$$\beta^+(v) > \beta^+(v - 1) .$$

The effect of increasing β^+ will be to increase the prices implied by (2.1) - (2.6) of the capital intensive goods relative to labour intensive goods. (Increasing β^+ increases the costs associated with using capital.) Hence, we can expect that by increasing β^+, we will switch demand and employment towards labour intensive industries, and thus decrease the size of the total capital stock. Investment in the snapshot year, which is related to capital stocks, is likely to be lower, i.e., increasing β^+ is likely to increase the consumption share of the GNP.

At present we have no experience with the adjustment rule (6.4). However, in Dixon and Butlin [1975], a model with a fixed β^+ and an endogenous savings share was solved. If the adjustment rule (6.4) caused difficulties, we could simply build up the schedule

$$s = f(\beta^+) ,$$

where s is the aggregate savings ratio.

Considerable experience has been accumulated with the iterative variable θ^+.[1] It is used to cope with the ad valorem tariffs. Normally we can expect the adjustment rule (6.5) to lead to rapid convergence. The shadow price, θ^* , on the balance of trade constraint (2.6), tends to be rather insensitive to changes in the value of θ^+, and hopefully to changes in the other iterative variables. Hence, $\theta^*(1)$ tends to be a good estimate of the final value of θ^*.

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[1] See, for example, Evans [1972], Dixon and Butlin [1975].

The final adjustment rule (6.6) is designed to force problem (2.1) - (2.6) to generate the appropriate distribution of expenditure across the consumer groups, (see (1.2)). w_i^+ is our estimate of the reciprocal of the marginal utility of expenditure in the SNAPSHOT solution for the i^{th} consumer group, (see (3.1)). Via (6.6) we update this estimate according to our latest estimate of prices, $p^*(v - 1)$, and the appropriate level for consumer i's spending, $(1 - s_i)\alpha_i(GNP^*(v - 1))$. The denominator of (6.6) merely imposes the normalization that $\sum_i w_i(v) = 1$.

Intuitively, (6.6) works as follows. If consumer i is spending too much, i.e.,

$$(p^*(v - 1))'C_i^*(v - 1) > (1 - s_i)\alpha_i(GNP^*(v - 1)) ,$$

then adjustment rule (6.6) will tend to lower w_i^+ , i.e., $w_i^+(v) < w_i^+(v - 1)$.[1] (6.6) seeks to decrease consumer i's expenditure by raising his marginal utility of expenditure (or decreasing its reciprocal). In practice, adjustment rules similar to (6.6) have been found to be very effective, see for example Dixon [1975 a, b].

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[1] It is possible that all or most consumers are spending too much (i.e., the overall saving level is too low). In this case, (6.6) will tend to lower the weights, w_i^+ , associated with consumers who are spending <u>relatively</u> too much, while the adjustment of β^+ will raise aggregate saving.

4. THE COMPUTATION OF SOLUTIONS FOR THE AUXILIARY PROGRAMMING PROBLEM

A key requirement for the success of the SNAPSHOT algorithm outlined in sections 2 and 3 is that problem (2.1) - (2.6) is easily computable. This problem must be solved at each stage of the iterative process (6.1) - (6.6).

In practice, it is convenient to solve (2.1) - (2.6) by considering the following closely related problem :

Choose X_1, X_2, C, M, all non-negative n-order vectors,

to maximize

$$\text{(2.1)(a)} \qquad U(w^+, C) \; - \; \theta^+ (\bar{p}^m)' (\hat{\bar{\tau}}) M$$

subject to

$$\text{(2.2)(a)} \qquad -(I - A)(X_1 + X_2) + K(\hat{\bar{r}}\beta^+ + \hat{\eta})X_2 + C + \bar{G} + \bar{E} - M + \chi^+ \leqslant 0 \; ,$$

and (2.3) - (2.6) ,

where

$$\text{(9.1)} \qquad U(w^+, C) = \max_{C_i} \left\{ \sum_i w_i^+ U_i(C_i) \,\middle|\, \sum_i C_i \leqslant C, \; C_i \geqslant 0 \forall i \right\} \; ,$$

i.e., $U(w^+, C)$ is the maximum value attainable for $\sum_i w_i^+ U_i(C_i)$, where the C_i are an allocation of the aggregate vector C across consumers. w^+ is the vector $(w_1^+, \ldots, w_m^+)$.

The advantage of the modified problem (2.1)(a), (2.2)(a), (2.3) - (2.6), henceforth MP, over the original problem is that we have reduced the number of variables. Whereas the original problem has nm consumption variables, the MP has only n. This reduction is particularly useful because the consumption variables provide the main difficulty in solving the auxiliary programming problem. They are the only variables in the non-linear part of the problem.

Proposition 4 establishes the relationship between the original and the modified problems.

Proposition 4:

Let $C_i(w^+, C)$, $i = 1, \ldots, m$ denote the[1] solution to the problem on the right side of (9.1), i.e., the $C_i(w^+, C)$, $i = 1, \ldots, m$ maximize $\sum_i w_i^+ U_i(C_i)$ subject to $\sum_i C_i \leqslant C$, $C_i \geqslant 0 \forall i$. Then

$S_1^* = \{X_1^*, X_2^*, C^*, M^*\}$ is a solution to the MP if and only if

$S_2^* = \{X_1^*, X_2^*, C_1(w^+, C^*), \ldots, C_m(w^+, C^*), M\}$ is a solution to (2.1) - (2.6).

Proof:

Assume S_1^* is a solution to the modified problem, but that S_2^* does not solve (2.1) - (2.6). Then there exists $S^{**} = \{X_1^{**}, X_2^{**}, C_1^{**}, \ldots, C_m^{**}, M^{**}\}$ satisfying the constraints (2.2) - (2.6) such that

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[1] The strict concavity of the U_i is sufficient to ensure that the $C_i(w^+, C)$ are unique.

$$\sum_i w_i^+ U_i(C_i^{**}) - \Theta^+(\bar{p}^m)'\hat{\tau}M^{**} > \sum_i w_i^+ U_i(C_i(w^+, C^*)) - \Theta^+(\bar{p}^m)'\hat{\tau}M^* .$$

We note that $U(w^+, C^*) = \sum_i w^+ U_i(C_i(w^+, C^*))$, and that $U(w_i^+, \sum_i C_i^{**}) \geqslant \sum_i w_i^+ U_i(C_i^{**})$. Therefore,

$$(9.2) \qquad U(w_i^+, \sum_i C_i^{**}) - \Theta^+(\bar{p}^m)'\hat{\tau}M^{**} > U(w^+, C^*) - \Theta^+(\bar{p}^m)'\hat{\tau}M^* .$$

Since $\{X_1^*, X_2^*, \sum_i C_i^{**}, M^{**}\}$ is a feasible solution for the modified problem, (9.2) is inconsistent with S_1^* being a solution to the modified problem.

Similarly, if we assume that S_2^* solves the original problem, but that S_1^* does not solve the MP, we can obtain a contradiction. This completes the proof of the proposition.

Proposition 4 implies that we may compute solutions for problem (2.1) - (2.6) in two steps. First, we solve the MP. Then we complete the solution of (2.1) - (2.6) by computing the $C_i(w^+, C^*)$, $i = 1, \ldots, m$, to maximize $\sum_i w_i^+ U_i(C_i)$ subject to $\sum_i C_i \leqslant C^*$, $C_i \geqslant 0$ for all i.

This two stage procedure is particularly easily implemented if (as is the usual case in applied work) the U_i are strictly concave additive functions. In that case, it can be shown that $U(w^+, C)$ is a strictly concave and additive function with respect to C,[1] i.e., there exist strictly concave (with respect to $C_{.j}$) functions V_j, $j = 1, \ldots, n$, such that

$$U(w^+, C) = \sum_j V_j(w^+, C_{.j}) \quad ,$$

where $C_{.j}$ is the j^{th} element of C.

The MP is then a very standard non-linear programming problem, concave, additive objective function and linear constraints.[2] Accurate approximate solutions may be obtained by solving the linear programming problem,

choose λ_{jt} , $j = 1, \ldots, n$, $t = 1, \ldots, T + 1$; X_1; X_2 and M, all non-negative, to maximize

$$(2.1)(b) \qquad \sum_{j=1}^{n} \sum_{t=1}^{T+1} \lambda_{jt} V_{jt} - \Theta^+ (\bar{p}^m)' (\hat{\bar{\tau}})M$$

subject to

$$(2.2)(b) \qquad -(I - A)(X_1 + X_2) + K(\hat{\bar{r}}\beta^+ + \hat{\eta})X_2 + H\lambda + \bar{G} + \bar{E} - M + \chi^+ \leq 0 \quad ,$$

(2.3) - (2.6) , and

$$(2.7) \qquad \sum_{t=1}^{T+1} \lambda_{jt} = 1 \quad , \qquad j = 1, \ldots, n \; ,$$

where H is an $n \times ((T+1)n)$ matrix of the form

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[1] See Dixon [1975a, pp. 68-72].

[2] See Hadley [1964, Chapters 7, 9].

$$\begin{bmatrix} H_{11} & \cdots & H_{1\ T+1}, & 0 & \cdots & 0 & , & 0 & \cdots & 0 \\ 0 & & 0 & , H_{21} & \cdots & H_{2\ T+1}, & & 0 & \cdots & 0 \\ & & & & & & & & & \\ 0 & & 0 & , 0 & \cdots & 0 & , & & & H_{n\ T+1} \end{bmatrix}$$

and the H_{jt} are parameters chosen so that we may assume that the solution of MP for $C_{.j}$ lies in the interval $[H_{j1}, H_{j\ T+1}]$ and that $H_{j1} < H_{j2} < \ldots < H_{j\ T+1}$. The V_{jt} are parameters whose values are determined by

$$V_{jt} = V_j(w^+, H_{jt})$$

and

$$\lambda' = (\lambda_{11}, \ldots\ldots\ldots\ldots \lambda_{n\ T+1}) .$$

Figure 2 illustrates the function V_j (solid curve) and the points $(H_{j1}, V_{j1}), \ldots, (H_{j\ T+1}, V_{j\ T+1})$ are marked for the case $T = 3$. V_j is approximated by joining these points by linear segments (broken line).

It is probably clear to the reader that the linear programming problem will generate approximate solutions to MP if we add the restrictions

(10.1)(a) No more than two of $\lambda_{j1}, \ldots, \lambda_{j\ T+1}$ can be non-zero ;

(10.1)(b) If two of the $\lambda_{j1}, \ldots, \lambda_{j\ T+1}$ are non-zero, then they must be adjacent, i.e., they must be a pair of the form $\lambda_{jt}, \lambda_{j\ t+1}$.

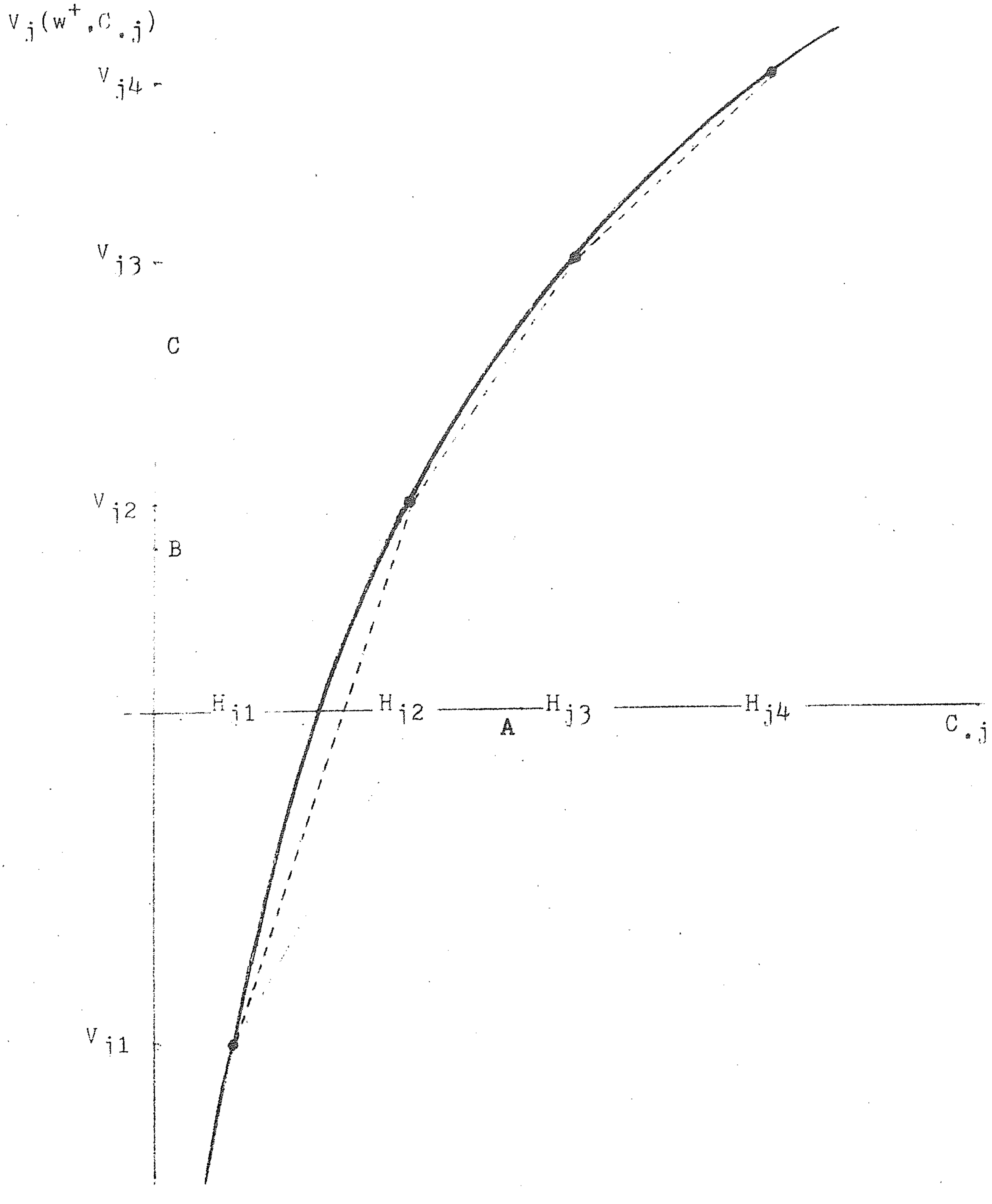

Figure 2

Fortunately, it is unnecessary to include (10.1)(a) and (b). Any solution to (2.1)(b), (2.2)(b), (2.3) - (2.7) necessarily satisfies them. Consider, for example, a case in which

$$(10.2) \qquad \lambda_{j1} = \tfrac{1}{2} , \qquad \lambda_{j4} = \tfrac{1}{2} .$$

$C_{.j}$ will be located at A (figure 2), i.e., $C_{.j} = \frac{1}{2} H_{j1} + \frac{1}{2} H_{j4}$. Utility resulting from (10.2) will be $B = \frac{1}{2} V_{j1} + \frac{1}{2} V_{j4}$. By replacing (10.2) by

$$\lambda_{j2} = \frac{H_{j3} - A}{H_{j3} - H_{j2}} , \qquad \lambda_{j3} = \frac{A - H_{j2}}{H_{j3} - H_{j2}}$$

we can increase the objective function (2.1)(b) by (C - B) and not violate the constraints. Hence, (10.2) is not possible as part of a solution to the linear programming problem. Following a similar argument, the reader can easily check that no more than two of the λ_{jt}, $t = 1, \ldots, T+1$, will ever be non-zero.

Two final points to be considered are - 1) the evaluation of the V_{jt}, or more generally, how do we evaluate $V_j(w^+, C_{.j})$, and 2) the choice of the H_{jt}. First, the V_{jt}.

Under the additivity assumption, the utility function of the i^{th} consumer can be written as

$$U_i(C_i) = \sum_j U_{ij}(C_{ij}) ,$$

where C_{ij} is the j^{th} element of C_i.

Then

$$(11.1) \qquad V_j(w^+, C_{.j}) = \max_{C_{ij}} \{\sum_i w_i^+ U_{ij}(C_{ij}) \mid \sum_i C_{ij} = C_{.j}, C_{ij} > 0 \forall i\} .$$

The C_{ij} on the right side of (11.1) can be evaluated by considering the system

$$(12.1) \qquad w_i^+ \frac{\partial U_{ij}(C_{ij})}{\partial C_{ij}} = \Gamma_j , \qquad i = 1, \ldots, m$$

$$(12.2) \qquad \sum_i C_{ij} = C_{.j} .$$

This system is easily handled. For example, we might use (12.1) to write $C_{ij} = f_{ij}(\Gamma_j)$, and then substitute into (12.2) giving $\sum_i f_{ij}(\Gamma_j) = C_{.j}$. Thus the problem is reduced to one equation with one unknown. Having computed the C_{ij}, we evaluate $V_j(w^+, C_{.j})$ by substituting the computed values into $\sum_i w_i^+ U_{ij}(C_{ij})$.

System (12.1) - (12.2) is also used in the second stage of our solution of the auxiliary programming problem. The first stage (i.e., the solution of MP via the "approximate" linear programming problem) generates values for the $C_{.j}^*$, $j = 1, \ldots, n$. Using (12.1) and (12.2) we can compute the C_{ij}^*, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

For the H_{jt} we can proceed as follows. Initially we may have little idea as to the eventual SNAPSHOT solution for the aggregate consumption of good j, $C_{.j}$. Hence we choose H_{j1} close to zero, $H_{j,T+1}$ very large, and space $H_{j2} \ldots H_{jT}$ evenly throughout the interval

$[H_{j1}, H_{j,T+1}]$. After our initial solution of the linear programming problem we will have an indication of what is the relevant range in which to look for $C_{.j}$. We reset the H_{jt} so that the function V_j is well approximated (by the broken line, figure 2) in the relevant range. A resetting rule that was used successfuly in Dixon [1975b] was :

(13.1) $H_{j1}, H_{j,T+1}$ fixed .

(13.2)
$$H_{j2}(v) = C^*_{.j}(v - 1) - |C^*_{.j}(v - 1) - C^*_{.j}(v - 2)| ,$$
$$H_{jT}(v) = C^*_{.j}(v - 1) + |C^*_{.j}(v - 1) - C^*_{.j}(v - 2)| ,$$
$$v = 3, 4 \ldots .$$

(13.3) $H_{jt}(v)$, $t = 3, \ldots, T-1$, are spaced evenly in the interval $[H_{j2}(v), H_{jT}(v)]$.

After the initial two solutions, the rules (13.1) - (13.3) have the effect of concentrating the H_{jt} around the most recently computed value for $C_{.j}$. Some simple modification is necessary if either $H_{j2}(v)$ or $H_{jT}(v)$, as computed from (13.2), is out of the interval $[H_{j1}, H_{j,T+1}]$, and where $C^*_{.j}(v - 1) = C^*_{.j}(v - 2)$.

5. CONCLUSION

In this paper, I have proposed a method for computing solutions for the SNAPSHOT model listed in table 1. The method consists of solving a sequence of non-linear programming problems - the auxiliary programming problems.

In section 2, I set out the auxiliary problem ((2.1) - (2.6)) and the necessary and sufficient conditions (3.1) - (3.9) for its solution. Then I proved propositions 1 and 2 which show that (a) if a solution to system (3.1) — (3.9) satisfies a list of additional conditions (4.1) - (4.8), then it reveals a SNAPSHOT solution via (5.1) - (5.20), and (b) every SNAPSHOT solution may be generated via (3.1) - (3.9).

In section 3, I suggested a set of rules ((6.1) - (6.6)) which are designed to adjust the parameters of the auxiliary programming problem so that the solution does satisfy the additional conditions (4.1) - (4.8). The reason for each of the adjustment rules was reviewed, and I proved proposition 3 which shows that the adjustments will only terminate when the solution to the auxiliary programming problem is consistent with (4.1) - (4.8). It is probably worth emphasising that an iterative procedure (of which (6.1) - (6.6) is an example) is in fact required. The reader may wonder why I did not simply write a programming problem which embodied the additional conditions (4.1) - (4.8). The reason is that such an approach would be no simpler than attempting direct methods to solve the potentially very large system of non-linear equations and inequalities (1.1) - (1.20).

Section 4 discusses solution procedures for the auxiliary programming problem. The success of our method of computing SNAPSHOT solutions depends on the auxiliary programming problem being "trivial." This appears to be the case, especially when the utility functions are additive. In a large SNAPSHOT model, we might have 100 commodities (the approximate size of the Australian input-output tables). Of these, perhaps 20 might enter the utility functions, and say 30 might be "significantly" imported. Experience suggests that T = 10, i.e., a 10 piece approximation to the functions $V_j(w^+, C_{.j})$ is more than adequate.[1] On these figures, the auxiliary programming problem can be handled by solving the 252 × 450 linear programming problem ((2.1)(b), (2.2)(b), (2.3) - (2.7)) (see table 2). Linear programs of this size are well inside the range of modern computers and standard LP packages. Also, the computer costs involved in solving a series of 252 × 450 LP's can be expected to be little more than the cost of solving the initial program. LP packages invariably have "basis" saving facilities, and the basis from the t^{th} LP solution will normally provide a very good starting point for the Simplex-search for the $(t+1)^{th}$ solution.

TABLE 2 : THE SIZE OF THE APPROXIMATE MODIFIED PROBLEM

Constraints		Variables	
(2.2)(b)	100	λ_{jt}	220
(2.3)	30	X_1	100
(2.4)	1	X_2	100
(2.5)	100	M	30
(2.6)	1		
(2.7)	20		
Total	252		450

No convergence propositions have been offered in this paper. This omission is easy to explain - the SNAPSHOT problem is too complicated and the adjustment rules (6.1) - (6.6) are too specific. It seems to me that even if it is possible to prove convergence propositions for our algorithm, the effort would be misplaced. Convergence propositions can merely suggest that an algorithm will work : there is almost always a rather large gap between applications and the assumptions of the

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1. See Dixon [1975b].

theoretical propositions.[1] For example, it is common to assume that iterative variables are adjusted "smoothly," whereas in practice large jumps are often made. Also, convergence propositions rarely have anything to say about the speed of convergence. Therefore, my view is that it is probably sensible only to devote scarce research time to convergence propositions in simplified prototype examples. Such propositions will indicate the conditions under which a particular algorithm is likely to be successful in full scale practical applications, and it is unlikely that one could gain much more than this by attempting propositions specific to our particular problem. Extensive theoretical work in simplified models has already been done, Dixon [1975a, chapter 3], Mantel [1971], and the success of the joint maximization approach in numerous tests (see Dixon [1975a, chapter 1], Dixon [1975b], Osterrieth and Waelbroeck [1975] and Ginsburgh and Waelbroeck [1974]) provides a reasonable basis for predicting success in the present application.

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[1] Perhaps an exception to this generalization is the work of Scarf [1973]. However, in practice, few researchers would wish to follow precisely the original Scarf procedure. Variations are known (but not proved) to speed convergence. See Merrill [1972] for a survey.

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