



IMPACT OF DEMOGRAPHIC CHANGE ON INDUSTRY STRUCTURE IN AUSTRALIA

A joint study by the Australian Bureau of Statistics, the Industries Assistance Commission, the Department of Labor and Immigration and the Department of Manufacturing Industry.

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THE SOLUTION METHOD
FOR THE ORANI MODULE

by

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Industries Assistance Commission

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ABSTRACT

This paper describes the method of solving the ORANI module of the IMPACT model. The structural equations of the ORANI module number tens of thousands and each is linear in the variables. The solution method is not only efficient in terms of computation, but also allows great flexibility in the choice as to which variables are endogenous and exogenous. This would enable the module to be used for many different types of study, especially in the policy arena. The first step in the solution is the elimination from the initial structural equations of all variables which are of no further interest. This results in the "condensed structural equations" which consist of eight matrix equations representing 600 single line equations in 1100 variables. The next step is to form the "final structural equations" by elimination of another 400 variables which will be "always endogenous" for all problems of interest. The "final structural equations" are then written as one large matrix equation. This makes it easy to obtain matrix expressions for the reduced form for any selection of the remaining variables as endogenous. The solution includes inversion of a matrix of dimension 200, and finally back substitution to obtain results for the 400 "always endogenous" variables which were previously eliminated.

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1. INTRODUCTION

This paper is concerned with the method of obtaining numerical solutions to the ORANI module of the IMPACT module.¹ The theoretical structure of the module and the fundamental equations have been described by Dixon.² The module is intended for use in a wide range of studies, and for this reason the aim is to devise a method of solution which is not only efficient but also extremely flexible. Rather than follow the usual procedure of solving the module on the basis of one particular classification of variables into endogenous and exogenous, great importance has been placed on flexibility in the choice of endogenous variables. In this way the module can be used to investigate the effects of many different kinds of policies by using different sets of endogenous variables. Further flexibility is achieved by providing for the addition of extra variables and equations in the final stages of the solution.

* I am grateful to Peter Dixon for many helpful discussions and explanations concerning the ORANI module, and also for useful suggestions about the method of solution. The material presented in appendix 1 is essentially a reworking of his unpublished notes. Alan Powell has made many helpful comments with regard to the method of presentation.

1. Alan A. Powell and Tony Lawson, "IMPACT : An Economic-Demographic Model of Australian Industry Structure - Preliminary Outline," Impact of Demographic Change on Industry Structure in Australia, Working Paper I-01, Industries Assistance Commission, Melbourne, September, 1975.
2. Peter B. Dixon, "The Theoretical Structure of the ORANI Module," Impact of Demographic Change on Industry Structure in Australia, Working Paper O-01, Industries Assistance Commission, Melbourne, October, 1975.

The structural equations of the ORANI module number tens of thousands. Each equation is linear in the variables, and each variable is the differential of a logarithm; i.e., each variable equals the proportional change in some quantity. Section 2 describes in general terms a method of obtaining reduced forms from such a large system. The method may have other applications, and it is described in such a way that it could be adapted to quite different models. It is necessary that each equation be linear in the variables.¹ However, there is no requirement that the variables be differentials of logarithms. The essential principles are the elimination of all variables which hold no further interest beyond their role in the initial structural specification, the (temporary) elimination of those variables which will be endogenous for all problems of interest, and the representation of n linear equations in matrix form in such a way that any n of the variables can be selected as endogenous. The first two principles produce an efficient solution, while the third enables the module to be used for many different studies. The succeeding sections describe the application of these principles to the ORANI module in particular.

In section 3 (and Appendix 1) it is described how the structural equations of the ORANI module can be condensed to eight matrix equations, or approximately 600 single line equations in 1100 variables, by the elimination

1. Some models which are linear in the variables are :

Leif Johansen, A Multisectoral Study of Economic Growth, 2nd ed., (Amsterdam : North Holland Publishing Company, 1974) ;

Lance Taylor and Stephen L. Black, "Practical General Equilibrium Estimation of Resource Pulls under Trade Liberalization," Journal of International Economics, Vol. 4 (1974), pp. 37-58 ;

Roger D. Norton, "A Computable General Equilibrium Framework for Structural Models of Economic Development," Prototype Country Models Working Note No. 3, Development Research Center, International Bank for Reconstruction and Development.

The variables are differentials of logarithms for the first two papers and for substantial parts of the third paper.

of tens of thousands of variables. In doing this it is assumed that it will never be necessary to exogenize or even to make further use of any of the eliminated variables. Section 4 describes how the system can be collapsed even further, resulting in a system of approximately 200 single line equations. Again it is assumed that, for the policies to be studied it will never be necessary to make any of the eliminated variables exogenous. Section 5 describes the determination of the reduced form for specified combinations of endogenous and exogenous variables. This involves inversion of a square matrix of dimension approximately 200, and then back substitution to obtain results for the variables which were eliminated in section 4.

An example of the use of the ORANI module would be to determine the effect of a certain wage settlement on employment in a particular occupation. Then the proportional change in the wage would be an exogenous variable and the proportional change in the level of employment would be an endogenous variable. On the other hand, if we wanted to know the wage level which would clear the labour market, then the proportional change in the wage would be endogenous and the proportional change in employment would be exogenous. The proportional change in the wage is an example of a variable being regarded as endogenous in one policy application and exogenous in another. As in this case, studies often occur in pairs which involve a simple swap of two variables between endogenous and exogenous.

A similar example concerns the relationship between the rate of tariff and the level of total domestic production in a particular industry. The effect of the tariff is to multiply the import price by some factor. This is equivalent to a proportional change in the import price. To determine the tariff necessary to achieve a particular level of production, the

proportional change in the level of production is treated as exogenous and the proportional change in the import price as endogenous. The opposite applies if the tariff is specified and the level of production is to be computed.

It should be kept in mind that in each of these examples there are many other variables, both endogenous and exogenous, in addition to those mentioned. The dependence of a particular endogenous variable on a particular exogenous variable will depend on which other variables are in the endogenous set.

2. GENERAL PRINCIPLES

The structural equations of an economic model such as the ORANI module specify relationships between a large number of variables. The variables frequently cover a wide spectrum of aggregation. In the ORANI module variables such as wages (by occupation), price, volume of exports, and levels of production (by input-output category) are of primary interest. They will be measured, estimated, and interpreted in the context of providing information to policy formation. On the other hand, the module also includes variables at a more disaggregated level such as the flow of goods from one particular industry to another. These variables play an important role in the structural equations, but apart from this they would rarely be of interest. There will be no occasion to supply them to the model as exogenous, and it is unlikely that there will ever be a need to deduce them from the model as endogenous. Henceforth we will refer to such variables as latent variables.

The problems of interest can be specified and solved without any reference at all to the latent variables. Consequently the latent variables can be eliminated from the structural equations, leading to a smaller system of equations which we will describe as the condensed structural equations. It is implicit in this elimination that all the latent variables are endogenous. Having reached this stage in the algebra, the initial structural equations and the latent variables can be forgotten. In the case of the ORANI module the initial structural equations number tens of thousands and the elimination of many variables, particularly those which describe flows in the input-output table, reduces the system to approximately 600 equations in approximately 1100 variables (the eight matrix equations of the condensed structure).

The condensed structural equations describe the same economic model as the initial structural equations, but in a far more compact form. Solutions of the condensed structural equations will be compatible with the economic model described by the initial structural equations. It is quite likely that the condensed structural equations will also be compatible with other economic models, but that is of no relevance to the present discussion.

The structural equations of the ORANI module comprise a system of equations which is linear in the variables. Consequently the condensed structural equations will also be linear in the variables. Let the number of variables and number of linearly independent equations in the condensed system be m and n respectively, where m is greater than n .

By the properties of such a system of equations, we can select any n of these variables and obtain expressions for them in terms of the remaining $(m - n)$. It is convenient to write the final result with the n variables on the left hand side of the equations and the $(m - n)$ variables on the right hand side. When the n left hand variables include only endogenous variables, and the $(m - n)$ right hand variables include only exogenous variables, such a solution is known as a reduced form of the problem. Clearly the reduced form will depend on the particular combination of n variables selected for the left hand side.

Models of this kind are usually presented in the following manner. First, there is a list of n structural equations. Then follows a list of definitions of parameters and variables. The variable list is typically broken into two parts - n variables which are said to be endogenous, while the remaining variables are specified as exogenous. The solution of the model then consists of expressing each of the variables on the endogenous list as a

function of the variables on the exogenous list. In contrast to this usual procedure, the solution method described in this paper does not depend on there being a single classification of variables as exogenous and endogenous. Instead, the ORANI solution recognizes that the model should be thought of as specifying the relationship between variables, and the particular classification of variables as endogenous or exogenous will depend on what the model is being used for. The user has the freedom to choose which are the endogenous variables; and the remaining variables are exogenous. This option enables the same model to be used to examine quite different objectives and policies simply by altering the set of endogenous variables. In some policy applications of a model, a particular variable will be regarded as endogenous while in others it will be treated as exogenous.

Given a set of endogenous variables, the module can be solved by writing the complete system of equations in matrix form and rearranging the order of the variables so that

$$(A \ B) \begin{pmatrix} y \\ z \end{pmatrix} = 0 ,$$

where

y is the vector of endogenous variables ,

z is the vector of exogenous variables, and

A and B are matrices with A square and of full rank.

Then

$$Ay + Bz = 0 ,$$

and hence

$$y = -A^{-1} Bz .$$

Although this is a general result which allows for an arbitrary choice of endogenous variables, it involves two computing tasks which are costly for a large system of equations - the inversion of matrix A and the

multiplication of two large matrices. In the ORANI module, the magnitude of the multiplication is somewhat reduced because large sections of matrix B are empty. In general, this is likely to be the case for any large system. The inversion of matrix A could be simplified by partitioning the matrix. However, the optimum partitioning would depend on the particular choice of endogenous variables. Such a procedure would have to be programmed separately for each choice of endogenous variables and in this way the formulation would lose the flexibility described above.

The number of operations in a matrix inversion varies essentially as n^3 , where n is the dimension of the matrix. Hence it is highly desirable to reduce the size of the matrix to be inverted. This can be achieved if it can be assumed that some variables will always be endogenous. These variables are then eliminated by elementary algebra, thus reducing the number of equations and consequently the size of the matrix to be inverted. The elimination process is worthwhile if the savings in the matrix inversion exceed the cost of the additional algebraic operations (elimination of endogenous variables, and also back substitution to solve for these eliminated variables). It is particularly worthwhile if the special structure of the equations permits an elimination procedure which does not involve complicated algebraic manipulations.

After elimination of the latent variables we can consider the remaining variables as being in three broad categories:

- (1) Variables which, for all problems of interest, are always endogenous (e.g., in ORANI, the price of capital goods) ;
- (2) Variables which, for all problems of interest, are always exogenous (e.g., in ORANI, the supply of land) ; and
- (3) Variables which are sometimes exogenous; i.e., which may be endogenous or exogenous, depending on the particular problem (e.g., in ORANI, wages) .

Such a classification is rather subjective, and it might be argued that odd cases can be found such that every variable might be endogenous or exogenous at some time or another. However, there are practical advantages to be obtained by ignoring the odd cases and using such a scheme. It will next be shown that by treating the always endogenous variables differently from the other variables, it is possible to solve the model in an efficient manner while at the same time retaining sufficient flexibility to study a wide range of problems.

It is possible to eliminate from the condensed structural equations all the always endogenous variables, thus reducing the number of equations and number of variables even further. The resulting equations will be referred to as the final system of (structural) equations and will include only variables which are either always exogenous or sometimes exogenous. The elimination of a variable does not mean that it is irrecoverably lost. Once a solution is obtained for the other variables it is possible to obtain an expression for an eliminated variable by means of back substitution.¹ The condensed structural equations are used in this way to solve for those eliminated variables which are of interest. Only endogenous variables can be eliminated. If an exogenous variable were eliminated then the remaining equations would contain more endogenous variables than the number of equations, and hence there would be an infinity of solutions. It would be impossible to determine the correct solution without going through the counter-productive step of resurrecting the eliminated exogenous variables.

Up to this point the solution is completely general, except for the constraint imposed by the decision that certain variables will always be endogenous. Finally, the solution for a particular problem or policy is obtained by specifying which are the endogenous variables, forming the corresponding matrices A and B, and solving as described above. In principle,

1. This applies to the latent variables as well as to the always endogenous variables.

any set of variables remaining in the final system of equations can be arbitrarily selected as endogenous. In practice there may be restrictions on which combinations of variables can be used as endogenous and exogenous. This will be discussed further in section 4 with regard to the ORANI module.

The method depends critically on the decision as to which variables are considered always endogenous, and are thus eliminated in the algebra. (In practice, this choice may include some consideration of the ease of eliminating variables and also the size of the final matrix to be inverted.) It follows from the way in which the variables have been classified that the final system of equations will contain a sufficient number of endogenous variables to accommodate all combinations of endogenous and exogenous variables which are of interest. Any other combinations are either incompatible with the initial structural specification of the model or with the assumptions concerning which variables are always endogenous.

In practical situations there are additional tricks that can be used to increase the flexibility and efficiency of the solution method. For example, we may wish to determine the effect of wage changes on the consumer price index. Even if the structural equations of the model contain no allowance for the consumer price index, such a problem can be dealt with by introducing a new variable and a new equation to describe and define the consumer price index, and by appending the new equation to the final system of equations. The procedure is simplest when the variables occurring in the new equation(s) do not include any of those previously eliminated as always endogenous. In any case, it is not necessary to rework all the algebra starting from the structural equations.

In the previous section, mention was made of two problems which involved wage rates and levels of employment in different industries. In one the wages were exogenous and employment was endogenous. In the other, employment was endogenous and wages were exogenous.

By treating these two problems as a pair, the algebraic solution to one can be used to give a simple solution to the other. The reduced form for the first problem, which concerns setting the wage rates exogenously, can be written in matrix form as

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} v \\ z \end{pmatrix}$$

where

v is the vector of proportional changes in wage rates ;

z is the vector of the other exogenous variables ;

u is the vector of proportional changes in the levels of employment ;

y is the vector of the other endogenous variables ;

vectors u and v have the same dimension; and

c_{11} , c_{12} , c_{21} , and c_{22} are matrices with c_{11} square, and the c 's are functions of parameters in the final structural equations.

Then

$$u = c_{11}v + c_{12}z$$

and

$$y = c_{21}v + c_{22}z$$

The reduced form for the second problem will have v and y on the left hand side, and u and z on the right hand side. By rearrangement of the expressions for u and y it is easily seen that the desired reduced form is

$$\begin{pmatrix} v \\ y \end{pmatrix} = \begin{pmatrix} c_{11}^{-1} & -c_{11}^{-1}c_{12} \\ c_{21}c_{11}^{-1} & c_{22} - c_{21}c_{11}^{-1}c_{12} \end{pmatrix} \begin{pmatrix} u \\ z \end{pmatrix}$$

Although these formulae were derived for the particular case of wages and employment levels, the results are general. Given the solution to one problem, the solution to a related problem (which involves swapping the roles of some variables between endogenous and exogenous) can thus be obtained by a more efficient method than re-solving the related problem. The new method involves manipulation of the c matrices, but it avoids the more costly step of recalculating new c matrices from the final structural coefficients. There is a considerable saving in computing, particularly if the number of variables involved in the swap is small compared with the total number of endogenous variables.

3. CONDENSED STRUCTURAL FORM

The complete ORANI module has been described by Dixon.¹ The structural equations which represent the module are summarized in appendix 4 of his paper. In this section, these equations are consolidated into eight matrix equations, otherwise known as the condensed structural equations, which form the starting point for the general algebraic solution to be described in sections 4 and 5. The consolidation is achieved by eliminating from the structural equations all latent variables, i.e., those variables which are of no further interest. The general principles of this process will be discussed in the text, but the details are to be found in appendix 1.

Although the module has already been fully documented by Dixon, the notation will be repeated here. This will enable the present paper, which is concerned only with the mechanics of solving the module, to be read independently from Dixon's paper, which is concerned with specifying the initial structural equations of the module on the basis of explicit assumptions about economic behaviour.

There are some differences between the two papers in the use of symbols to represent variables. Attention will be drawn to these differences as they arise. In general, each variable will be defined only once, when it is first used. The variables remaining in the eight matrix equations of the condensed structure are listed in Table 4. Following Dixon, capital letters denote annual flows in real terms and small letters denote

proportional changes. Thus $x_i = \frac{\Delta X_i}{X_i}$. The variables

1. Peter B. Dixon, op. cit.

in the system of equations are all of the proportional change type. To avoid needless repetition in describing the variables, the phrase "the proportional change in" will usually be omitted in definitions.

The following conventions for subscripts and superscripts are adhered to as far as possible. For $x_{isj}^{(k)}$, the superscript (k) denotes usage as follows:

- k = 1 intermediate usage ;
- k = 2 new investment in capital goods ;
- k = 3 household consumption ;
- k = 4 exports ;
- k = 5 other uses (including government).

The subscript i denotes the input, as follows:

- i = 1 to g for goods, where g is the number of industries ;
- i = g + 1 for primary factors (wages, fixed capital, land) ;
- i = g + 2 for other costs of production (including taxes, subsidies, inventories and costs of holding liquidity).

The subscript s denotes the source of the input, as follows:

- for i = 1 to g $\left\{ \begin{array}{l} s = 1 \text{ for domestic sources} , \\ s = 2 \text{ for imports} ; \end{array} \right.$
- for i = g + 1 $\left\{ \begin{array}{l} s = 1(\ell) \text{ for labour in occupation } \ell \text{ } (\ell = 1, \dots, h) , \\ s = 2 \text{ for capital stock} , \\ s = 3(r) \text{ for land of type } r \text{ } (r = 1, \dots, n) . \end{array} \right.$

The subscript j denotes the using industry. It takes values 1 to g for intermediate usage (k = 1) and capital investment (k = 2). For k = 3, 4 or 5, the j subscript vanishes. Thus, for example, $x_{i2j}^{(2)}$ is the proportional change in the input of imports of good i into capital investment for industry j. p_{i2} is the proportional change in the import price of goods

of type i , whilst X_i is the total usage of goods of type i from all sources. When these quantities are expressed as vectors the subscript i is dropped. Hence p_2 is the vector of proportional changes in import prices.

These conventions may appear to be broken for the variables related to primary factors and other costs, i.e., q_{1l} , q_{2i} , q_{3r} , q_{4i} , η_{1l} , η_{2i} , η_{3r} and the related vectors q_1 , q_2 , etc.. This is because the first subscript, $(g+1)$ or $(g+2)$, has been dropped as too cumbersome. This should not cause any confusion as the meanings of the subscripts are fairly obvious from the contexts in which they are used.

In describing the matrix equations, use is often made of the operator $\hat{}$ above a variable to denote conversion of a vector to a diagonal matrix. Use is also made of I_{ij} to represent the elements of the identity matrix. Thus I_{ij} equals 1 for diagonal elements ($i=j$) and 0 for all other elements ($i \neq j$).

There are g types of good, h occupations (types of labour) and n types of land. Dimensions of matrices will not usually be stated explicitly. They can be deduced from the dimensions of vector variables in the matrix equations. Similarly, ranges of values for subscripts are not always stated as they can be inferred from the context.

The structural equations described by Dixon allowed for only one occupation and one type of land. They have now been changed to allow for h occupations and n types of land. The production technology allows for substitution between inputs of labour and land. However, it is assumed that there is no substitution between inputs of labour from different occupations and no substitution between inputs of land of different types. This means that for each industry there are fixed ratios between the numbers of workers

in each occupation, and fixed ratios between the numbers of units of land of each type. It is also assumed that the wage rate for a particular occupation is the same for all industries, and the rental price for a particular type of land is the same for all industries.

The structural equations now consist of the set of equations summarized by Dixon in Appendix 4 of his paper, together with the following changes:¹

(a) new equations which replace original equations

$$P_{(g+1)1j} = \sum_{\ell} S_{(g+1)1(\ell)j} q_{1\ell} \quad (6.3)$$

$$P_{(g+1)3j} = \sum_r S_{(g+1)3(r)j} q_{3r} \quad (6.5)$$

$$\eta_{1\ell} = \sum_j L_{\ell j} x_{(g+1)1(\ell)j} \quad (6.7)$$

$$\eta_{3r} = \sum_j R_{rj} x_{(g+1)3(r)j} \quad (6.9)$$

(b) new equations

$$x_{(g+1)1(\ell)j} = x_{(g+1)1j} \quad (3.6(iii))$$

$$x_{(g+1)3(r)j} = x_{(g+1)3j} \quad (3.6(iv))$$

(c) additional equations from the text of Dixon's paper : (3.14), (3.15), and (3.16).

1. The equations are numbered in the system used by Dixon.

Table 1 summarizes the relationships between this set of equations and the matrix equations. It should be pointed out that no use has yet been made of equations (3.6(i)), (3.10) and (3.17) for $s = 2$ (demands for imported goods). These are not required for the initial applications of the module. They will be introduced at some later date when an equation for the balance of trade is added.

Table 1 : Correspondence Between the Matrix Equations of the Condensed Structure and the Initial Structural Equations

Matrix Equations of the Condensed Structure	Initial Structural Equations	
	Basic Equations	Subsidiary Equations
1. Price of Capital Goods	(5.4)	
2. Domestic Commodity Prices	(5.3)	(6.3), (6.5)
3. Investment	(4.8)	(4.9), (4.10)
4. Household Demands	(3.17) (for $s=1$)	(3.14), (3.15), (3.16)
5. Labour Demands	(6.7)	(3.6(ii)), (3.6(iii)), (6.3), (6.5)
6. Demands for Capital	(6.8)	(3.6(ii)), (6.3), (6.5)
7. Demands for Land	(6.9)	(3.6(ii)), (3.6(iv)), (6.3), (6.5)
8. Market Clearing	(6.6)	(3.6(i)) (for $s=1$), (3.10) (for $s=1$)

Many of the structural equations involve parameters which must be supplied from outside the model. These parameters are summarized in Table 2. In addition, many of the coefficients in the equations are described as "shares" and are ratios of entries in the input-output table. For the purpose of this paper it is assumed that the table is in the form shown in figure 1.

Table 2 : Parameters to be Supplied to the ORANI Module

Parameter	Dimension	Description
$\sigma_{ij}^{(1)}$	$g \times g$	elasticity of substitution between domestic and imported sources of goods for intermediate usage
$\sigma_{ij}^{(2)}$	$g \times g$	elasticity of substitution between domestic and imported sources of goods for investment purposes
$\sigma_i^{(3)}$	$g \times 1$	elasticity of substitution between domestic and imported sources of goods for household consumption
α_i	$g \times 1$	elasticity of substitution between primary factors in the production process
β_j	$g \times 1$	elasticity of rate of return with respect to level of capital stock
Q_j	$g \times 1$	ratio of gross rate of return (without allowance for depreciation) to net rate of return
g_j	$g \times 1$	rate of growth of capital stock
η_{ij}	$g \times g$	elasticity of household demand with respect to prices
ϵ_i	$g \times 1$	elasticity of household demand with respect to total household expenditure
L_{lj}	$h \times g$	(shares derived from) matrix of employment by occupation and industry
R_{rj}	$n \times g$	(shares derived from) matrix of land use by land type and industry

Figure 1 : Format of Input-Output Table for the ORANI Module

Annual Flows

(\$ million)

	Intermediate Usage k = (1) j=1 to g	Capital Investment k = (2) j=1 to g	Household Consumption k = (3)	Exports k = (4)	Other k = (5)
Domestically Produced Goods s = 1 i=1 to g					
INPUTS OF GOODS					
Imported Goods s = 2 i=1 to g					
PRIMARY FACTORS i=g + 1					
Wages s = 1 h occupations					
Capital Stock s = 2					
Land s = 3 n types of land					
OTHER COSTS OF PRODUCTION i=g + 2					

The method of transforming the Australian Input-Output Tables (1962/63)¹ to this form will be described in a later paper. For the remainder of this paper the term "input-output table" refers to a table in the format of figure 1.

A mechanical approach to the table would sometimes indicate that all shares of some quantity are zero divided by zero. One convention would be to make all such shares zero. However, in order to preserve the condition that the sum of shares is unity, we have arbitrarily made one of the shares unity and the others zero. Detailed inspection of the algebra shows that these choices have no effects on the solution.²

Before turning to the detailed derivation of the eight matrix equations, it is worthwhile to make some general remarks about these equations. By way of example, consider the vector of household consumption of domestically produced goods ($X_{il}^{(3)}$) which is assumed to depend only on the vectors of domestic prices (P_{j1}) and import prices (P_{j2}), and total consumption expenditure (C). This relationship can be written in general terms as

$$X_{il}^{(3)} = F_{il}(P_{11} \dots P_{g1}; P_{12} \dots P_{g2}; C).$$

In differential form, this becomes

$$\frac{dX_{il}^{(3)}}{X_{il}^{(3)}} = \sum_j \frac{\partial F_{il}}{\partial P_{j1}} \frac{P_{j1}}{X_{il}^{(3)}} \frac{dP_{j1}}{P_{j1}} + \sum_j \frac{\partial F_{il}}{\partial P_{j2}} \frac{P_{j2}}{X_{il}^{(3)}} \frac{dP_{j2}}{P_{j2}} + \frac{\partial F_{il}}{\partial C} \frac{C}{X_{il}^{(3)}} \frac{dC}{C}$$

or

$$X_{il}^{(3)} = \sum_j \eta_{ilj1} P_{j1} + \sum_j \eta_{ilj2} P_{j2} + \epsilon_{il} C,$$

where

$$\eta_{iljs} = \frac{\partial F_{il}}{\partial P_{js}} \frac{P_{js}}{X_{il}^{(3)}} = \frac{\partial X_{il}^{(3)}}{\partial P_{js}} \frac{P_{js}}{X_{il}^{(3)}} \quad \text{and} \quad \epsilon_{il} = \frac{\partial X_{il}^{(3)}}{\partial C} \frac{C}{X_{il}^{(3)}};$$

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1. Australian National Accounts, Input-Output Tables 1962-63, Commonwealth Bureau of Census and Statistics, Canberra, May, 1973.
 2. There is one exception, in the demand equation for capital. None of the elements of S_2 should be set equal to 1.0 arbitrarily because this can lead to complete rows and columns of zeros in the system of equations and hence a matrix that cannot be inverted. For industries which use no capital stock, non-zero elements in S_1 and S_3 can cause curious dependencies of capital stock on wages and land rentals; however, such elements have no effect on other industries.

i.e., η_{ilj1} is the elasticity of consumption $X_{il}^{(3)}$ with respect to domestic price P_{j1} , η_{ilj2} is the elasticity of $X_{il}^{(3)}$ with respect to P_{j2} , and ϵ_{il} is the elasticity of $X_{il}^{(3)}$ with respect to C . This expression for $X_{il}^{(3)}$ is equivalent to Dixon's equation (3.17). In matrix form it becomes $x_1^{(3)} = \eta_{11}p_1 + \eta_{12}p_2 + \epsilon c$.

Each of the eight matrix equations can be interpreted in a similar manner. Given that a left hand variable is a function of right hand variables, the proportional change in the left hand variable is equal to the sum of the elasticities with respect to the right hand variables, each elasticity being multiplied by the proportional change in the same right hand variable. The role of the structural equations is to specify particular forms for these elasticities as functions of the initial structural parameters. However, it should be pointed out that even if the initial structural equations are changed, the general forms of equations 1 to 8 will be unchanged provided that each variable on the left hand side is still a function of the same variables on the right hand side. For example, the specification of investment might be changed but, providing investment was still a function only of the level of capital stock, the rental value of capital, the price of capital, and aggregate investment, equation (3) would still be of the form

$i = G_2\eta_2 + G_3q_2 + G_4\pi + G_5\theta$. The formulae for the elements of G_2 , G_3 , G_4 and G_5 in terms of initial structural parameters would be changed, but the algebra of the condensed structure would be the same.

The derivations of the eight matrix equations from the initial structural equations are described in appendix 1. Many of the eliminated variables correspond to proportional changes in flows in the input-output table. These include demands for goods for use in each of production,

creation of capital, and household consumption. For example, demands for domestic goods for use in production $(x_{ij}^{(1)})$ and capital creation $(x_{ij}^{(2)})$ are eliminated in the formation of equation (8). This accounts for the removal of $2g^2$ or 18,432 variables. Other variables are omitted as a consequence of technological assumptions such as the lack of substitution between labour from different occupations. The eight matrix equations are presented in Table 3, and the notation used in these equations is summarized in Table 4.

Table 3 : Condensed Structural Equations of the ORANI Module*

Equation 1 : Price of Capital Goods (g components)

$$\pi = A_1^{(2)} p_1 + A_2^{(2)} p_2 .$$

Equation 2 : Domestic Commodity Prices (g components)

$$p_1 = A_1^{(1)} p_1 + A_2^{(1)} p_2 + A_3^{(1)} q_1 + A_4^{(1)} q_2 + A_5^{(1)} q_3 + A_6^{(1)} q_4 .$$

Equation 3 : Investment (g components)

$$i = G_2 \eta_2 + G_3 q_2 + G_4 \pi + G_5 \theta .$$

Equation 4 : Household Demands (g components)

$$x_1^{(3)} = \eta_{11} p_1 + \eta_{12} p_2 + \epsilon c .$$

Equations 5 to 7 : Factor Demands

Labour (h components): $\eta_1 = L_0 x + L_1 q_1 + L_2 q_2 + L_3 q_3 ;$

Capital (g components) : $\eta_2 = K_0 x + K_1 q_1 + K_2 q_2 + K_3 q_3 ;$

Land (n components): $\eta_3 = N_0 x + N_1 q_1 + N_2 q_2 + N_3 q_3 .$

Equation 8 : Market Clearing (g components)

$$x = \beta^{(1)} x + \beta^{(2)} i + \beta^{(3)} x_1^{(3)} + \beta^{(4)} x_1^{(4)} + \beta^{(5)} x_1^{(5)} - \psi p_1 + \psi p_2 .$$

* For notation, see Table 4

Table 4 : Notation

Variable	Dimension	Equations	Description
π	g	(1), (3)	price of new capital (assumed always endogenous, eliminated in algebra)
p_1	g	(1), (2), (4), (8)	price of domestic goods (see m_1 and m_2 below)
p_2	g	(1), (2), (4), (8)	price of imported goods
q_1	h	(2), (5), (6), (7)	annual wages, i.e., price of labour by occupation
q_2	g	(2), (3), (5), (6), (7)	annual rental value of capital
q_3	n	(2), (5), (6), (7)	annual rental value of land
q_4	g	(2)	cost of extras (liquidity, inventories, taxes)
$i (\equiv y)$	g	(3), (8)	investment (assumed always endogenous, eliminated in algebra)
θ	1	(3)	aggregate investment (from MACRO)
c	1	(4)	aggregate consumption (from MACRO)
η_1	h	(5)	supplies of labour (from BACHUROO)
$\eta_2 (\equiv k)$	g	(3), (6)	capital stock
η_3	n	(7)	supply of land
x	g	(5), (6), (7), (8)	production, total output
$x_1^{(3)}$	g	(4), (8)	consumption (assumed always endogenous, eliminated in algebra)
$x_1^{(4)}$	g	(8)	exports (see m_1 and m_2 below)
$x_1^{(5)}$	g	(8)	other demands (government, inventory changes)
m_1	g		mixed vector of p_1 and $x_1^{(4)}$ (exogenous components)
m_2	g		mixed vector of p_1 and $x_1^{(4)}$ (endogenous components)
			g = number of industries (= 96, initially)
			h = number of occupations (= 8, initially)
			n = number of types of land (= 1, initially)

N.B. All variables are fractional changes, i.e., dimensionless quantities of the form $x = \frac{\Delta X}{X}$.

4. FINAL STRUCTURAL FORM

The eight matrix equations of the condensed structure consist of $(6g + h + n)$ equations which are linear in $(11g + 2h + 2n + 2)$ variables.¹ This means that we could select any $(6g + h + n)$ variables and express them in terms of the other $(5g + h + n + 2)$ variables. Stated another way, given the values of any $(5g + h + n + 2)$ variables as exogenous, we could compute the values of the remaining $(6g + h + n)$ variables as endogenous.²

For the initial trial version of the ORANI module with $g = 96$, $h = 8$ and $n = 1$, there are 585 equations in 1076 variables. This is still too large a system of equations for convenient computation. Following the strategy described in section 2, the aim is to reduce the number of equations and variables further by eliminating variables which will be endogenous for all problems of interest. The choice of such variables is somewhat subjective as it depends on anticipating all possible uses of the module. It is important to retain a sufficient number of variables so that there is adequate flexibility in the choice of endogenous and exogenous variables. From a computational point of view, some consideration was given to the ease of eliminating variables and also to the size of the final matrix to be inverted. It was decided to eliminate $4g$ variables, thus reducing the system to $(2g + h + n)$ equations in $(7g + 2h + 2n + 2)$ variables.

-
1. The economics of the model suggest that the equations are linearly independent.
 2. In practice, due to the particular form of the equations there are restrictions on the choice of endogenous variables. For example, the variables $x_1^{(4)}$ and $x_1^{(5)}$, each with g components, appear only in equation (8). It would be impossible to make all components of both $x_1^{(4)}$ and $x_1^{(5)}$ endogenous for, supposing that all other variables in equation (8) were either exogenous or already eliminated, there would be g linear equations in $2g$ unknowns. Such a system has an infinite number of solutions because any g of the variables can be given arbitrary values and these then determine the values of the other g .

It is assumed that the vectors π (price of new capital), i (investment) and $x_1^{(3)}$ (household consumption) are always endogenous. These account for $3g$ of the $4g$ variables to be eliminated. The remaining g always endogenous variables are a combination of components of the vectors p_1 (domestic prices) and $x_1^{(4)}$ (exports). It is assumed that each industry can be classified as either

- (a) an export industry, which has exogenously specified domestic prices (equal to world prices) and endogenously determined exports (excess production, as given by the model); or
- (b) a non-exporting industry, which has endogenous domestic prices (determined by the model) and exogenous levels of exports (usually zero).

It is intended that the number of export industries be no more than 20. The vectors p_1 and $x_1^{(4)}$, each with g components, between them contain g endogenous components and g exogenous components. These $2g$ components can be rearranged to form two new vectors, m_1 and m_2 , each with g components, one for each industry, where m_1 is the vector of the exogenous components of p_1 and $x_1^{(4)}$, and m_2 is the vector of the endogenous components of p_1 and $x_1^{(4)}$.

The elimination of π , i and $x_1^{(3)}$ from equations (1), (3), (4) and (8) is trivial and results in a new equation, (1b). The subsequent elimination of m_2 from equations (1b) and (2) is more complicated and results in a selection of equations from the two new matrix equations (1c) and (2a). The derivations of equations (1b), (1c) and (2a) are now described.

Derivation of Equation (1b)

Elimination of π , $x_1^{(3)}$, i from equations (1), (3), (4) and (8) gives equation (1b):

$$\begin{aligned} \beta^{(9)}x + \beta^{(2)}G_2n_2 + \beta^{(2)}G_3q_2 + D_1p_1 + D_2p_2 + \beta^{(2)}G_5\theta \\ + \beta^{(3)}\epsilon c + \beta^{(4)}x_1^{(4)} + \beta^{(5)}x_1^{(5)} = 0 \end{aligned} \quad (1b)$$

where

$$\begin{aligned} \beta^{(9)} &= -I + \beta^{(1)} , \\ D_1 &= \beta^{(2)}G_4A_1^{(2)} + \beta^{(3)}\eta_{11} - \psi , \\ D_2 &= \beta^{(2)}G_4A_2^{(2)} + \beta^{(3)}\eta_{12} + \psi . \end{aligned}$$

The elimination of 3g variables reduces the number of equations from $(6g + h + n)$ to $(3g + h + n)$.

Equation (1b) is reached by the following three steps :

- (i) Substitute equation (1) into equation (3) to eliminate π , giving equation (1a):

$$i = G_2n_2 + G_3q_2 + G_4A_1^{(2)}p_1 + G_4A_2^{(2)}p_2 + G_5\theta ; \quad (1a)$$

- (ii) Substitute equation (4) into equation (8) to eliminate $x_1^{(3)}$, giving equation (4a) :

$$\begin{aligned} 0 = \beta^{(9)}x + \beta^{(2)}i + \beta^{(3)}\eta_{11}p_1 + \beta^{(3)}\eta_{12}p_2 + \beta^{(3)}\epsilon c \\ + \beta^{(4)}x_1^{(4)} + \beta^{(5)}x_1^{(5)} - \psi p_1 + \psi p_2 ; \end{aligned} \quad (4a)$$

(iii) Substitute equation (1a) into equation (4a) to eliminate i , giving

$$\begin{aligned}
 0 = & \beta^{(9)}x + \beta^{(2)}G_2n_2 + \beta^{(2)}G_3q_2 + \beta^{(2)}G_4A_1^{(2)}p_1 + \beta^{(2)}G_4A_2^{(2)}p_2 \\
 & + \beta^{(2)}G_5\theta + \beta^{(3)}n_{11}p_1 + \beta^{(3)}n_{12}p_2 + \beta^{(3)}\epsilon c \\
 & + \beta^{(4)}x_1^{(4)} + \beta^{(5)}x_1^{(5)} - \psi p_1 + \psi p_2 .
 \end{aligned}$$

This simplifies to equation (1b) above.

Derivation of Equations (1c) and (2a)

Elimination of m_2 from equations (1b) and (2) will reduce the numbers of equations and variables by g to $(2g + h + n)$.

Equation (2) can be rewritten as

$$p_1 = D_4 \{ A_2^{(1)} p_2 + A_3^{(1)} q_1 + A_4^{(1)} q_2 + A_5^{(1)} q_3 + A_6^{(1)} q_4 \} \quad (2a)$$

$$\text{where } D_4 = (I - A_1^{(1)})^{-1} .$$

As all the elements of $A_1^{(1)}$ are non-negative and strictly less than 1.0, $(I - A_1^{(1)})$ can always be inverted.

Substitution of equation (2a) into equation (1b) to eliminate

p_1 gives

$$\begin{aligned}
 & \beta^{(9)}x + \beta^{(4)}x_1^{(4)} + \beta^{(5)}x_1^{(5)} + D_5 A_3^{(1)} q_1 + (\beta^{(2)}G_3 + D_5 A_4^{(1)}) q_2 + D_5 A_5^{(1)} q_3 \\
 & + D_5 A_6^{(1)} q_4 + (D_2 + D_5 A_2^{(1)}) p_2 + \beta^{(2)}G_2 n_2 + \beta^{(2)}G_5 \theta + \beta^{(3)}\epsilon c = 0 , \quad (1c)
 \end{aligned}$$

$$\text{where } D_5 = D_1 D_4 .$$

Equation (1c) can be written with the term $\beta^{(4)} x_1^{(4)}$ on the left hand side and all other terms on the right hand side. Then equations (2a) and (1c) together consist of $2g$ single line equations, one for each of the components of the vectors p_1 and $x_1^{(4)}$. The vectors p_1 and $x_1^{(4)}$ do not appear anywhere else in the remaining equations of the structure (5), (6) and (7). Hence m_2 , the g endogenous components of p_1 and $x_1^{(4)}$, can be eliminated from the structure simply by crossing out the corresponding lines in (2a) and (1c).

In practice, all elements of the matrices are calculated for both equation (2a) and equation (1c). The export industries are specified, and then matrices for the composite equation (1d) are formed from the matrices of (2a) and (1c). The i^{th} row will be taken from equation (2a) if the i^{th} industry is an export industry; otherwise it will be taken from equation (1c).

Equation (1d) has g components and consists of linear terms in the variables x , $x_1^{(5)}$, p_2 , q_1 , q_2 , q_3 , q_4 , n_1 , n_2 , n_3 , θ , c , and m_1 (parts of p_1 and $x_1^{(4)}$). It must be emphasized that the components of p_1 and $x_1^{(4)}$ remaining in the final structural equations are necessarily exogenous.

The final system of structural equations consists of equation (1d) (parts of (2a) and (1c)) together with

$$L_0 x + L_1 q_1 + L_2 q_2 + L_3 q_3 - I n_1 = 0; \quad (5)$$

$$K_0 x + K_1 q_1 + K_2 q_2 + K_3 q_3 - I n_2 = 0; \quad (6)$$

$$N_0 x + N_1 q_1 + N_2 q_2 + N_3 q_3 - I n_3 = 0. \quad (7)$$

The final system of equations is represented diagrammatically in figure 2. It is obvious from the figure that it would be easy to change the final system of equations to include all lines of equations (2a) and (1c). The final system of equations would then consist of approximately 300 equations rather than 200. This would be the case if it were decided that all components of π , $x_1^{(3)}$ and i were always endogenous, but none of the components of p_1 and $x_1^{(4)}$ were necessarily endogenous.

Figure 2 : Diagrammatic Representation of the Final Structural System of $(2g + h + n)$ Equations

Matrices without a symbol are full of zeros.

Cross-hatching indicates schematically the parts of matrices which are not included in the final system due to the distinction between export industries and non-exporting industries.

Eqn.	x	$x_1^{(4)}$	$x_1^{(5)}$	P ₁	P ₂	q ₁	q ₂	q ₃	q ₄	η_1	η_2	η_3	θ	c
2a	$\beta^{(9)}$	$\beta^{(4)}$	$\beta^{(5)}$	- I	$D_{42}^{A(1)}$	$D_{43}^{A(1)}$	$D_{44}^{A(1)}$	$D_{45}^{A(1)}$	$D_{46}^{A(1)}$					
1c					$D_{52}^{A(1)} + D_2$	$D_{53}^{A(1)}$	$D_{54}^{A(1)} + \beta^{(2)}G_3$	$D_{55}^{A(1)}$	$D_{56}^{A(1)}$		$\beta^{(2)}G_2$		$\beta^{(2)}G_5$	$\beta^{(2)}\epsilon$
5	L_0					L_1	L_2	L_3		- I				
6	K_0					K_1	K_2	K_3			- I			
7	N_0					N_1	N_2	N_3				- I		

5. REDUCED FORM5.1 Impact Elasticities

The final system of $(2g + h + n)$ structural equations can be written

$$Ay + Bz = 0 ,$$

where y is the vector of the $(2g + h + n)$ endogenous variables and z is the vector of the $(5g + h + n + 2)$ exogenous variables.

Then

$$y = Cz , \text{ where } C = -A^{-1}B .$$

The initial aim is not to obtain actual numerical values of the endogenous variables for given values of the exogenous variables, but rather to determine impact elasticities, i.e., the percentage changes in endogenous variables that arise from a one percent change in an exogenous variable, with all other exogenous variables held constant. The equation $y = Cz$ can be written in terms of individual variables as

$$y_i = \sum_j C_{ij} z_j$$

$$\text{i.e., } \left(\frac{\Delta Y_i}{Y_i}\right) = \sum_j C_{ij} \left(\frac{\Delta Z_j}{Z_j}\right)$$

$$\therefore C_{ij} = \left(\frac{\partial Y_i}{\partial Z_j}\right) \frac{Z_j}{Y_i} .$$

C_{ij} is thus the elasticity of the endogenous variable Y_i with respect to the exogenous variable Z_j . Hence C is a matrix of the desired impact elasticities.

5.2 Back Substitution

It is necessary to find corresponding results for the other endogenous variables ($x_1^{(3)}$, i , π and m_2 [the endogenous components of p_1 and $x_1^{(4)}$]).

First consider m_2 . Using those rows of equations (2a) and (1c) which correspond to the endogenous components of p_1 and $x_1^{(4)}$, m_2 can be written in the form

$$\begin{aligned} m_2 &= M_1 y + M_2 z \\ &= (M_1 C + M_2) z \\ &= M_3 z . \end{aligned}$$

π depends on p_1 and p_2 as given by equation (1).

The components of p_2 are either exogenous or are given by $y = Cz$. Some of the components of p_1 are exogenous, and the remainder are given by m_2 .

Hence, from equation (1), π can be written in the form

$$\begin{aligned} \pi &= Q_1 y + Q_2 z \\ &= (Q_1 C + Q_2) z \\ &= Q_3 z . \end{aligned}$$

Similarly, from equation (4)

$$\begin{aligned} x_1^{(3)} &= R_1 y + R_2 z \\ &= R_3 z , \quad \text{where } R_3 = R_1 C + R_2 . \end{aligned}$$

Finally, from equation (3), i can be written in the form

$$\begin{aligned} i &= S_1 y + S_2 z + G_4 \pi \\ &= S_1 C z + S_2 z + G_4 Q_3 z \\ &= (S_1 C + S_2 + G_4 Q_3) z \\ &= S_3 z . \end{aligned}$$

For the initial solution, the matrix C is of dimension 201×491 , or approximately 100,000 elements.

The matrices M_3 , Q_3 , R_3 and S_3 are each of the dimension 96×491 , or approximately 50,000 elements each.

The evaluations of M_3 , Q_3 etc. involve the multiplication of very large matrices. However, some computational savings can be achieved by taking advantage of the many columns of zeros that exist in matrices such as Q_1 and Q_2 .

5.3 The Computer Program

The fundamental computer program for the ORANI module uses as data the matrices of equations (1c), (2a), (5), (6) and (7) (see figure 2 for a diagrammatic representation). To these the user supplies

- (i) a list of "export industries" (i.e., industries for which the domestic price is exogenous), which determines the composition of m_2 and hence the components of equations (1c) and (2a) to be used ;
- (ii) a complete list of $(2g + h + n)$ endogenous variables¹ - for each vector it is possible to specify either the entire vector or only selected components as endogenous ;
- (iii) the combinations of exogenous and endogenous variables for which impact elasticities are required.

In its present form the program does not use inputs of exogenous variables. The initial aim is to determine elasticities. The impact elasticities for the final $(2g + h + n)$ endogenous variables are

1. As discussed in section 4, there are restrictions on the choice of endogenous variables. For example, all components of p_1 and $x_1^{(4)}$ remaining in the final equations are necessarily exogenous. Also, by the footnote on the first page of section 4, since most components of $x_1^{(4)}$ are assumed always endogenous, most components of $x_1^{(5)}$ are necessarily exogenous.

obtained from the final system of equations and involve inversion of a square matrix of dimension $(2g + h + n)$.¹ The impact elasticities for the eliminated variables are then obtained by back substitution in the appropriate equations. The inverted matrix and the lists (i) and (ii) are saved. Thus, for the same selection of export industries and endogenous variables, it is a simple matter to output impact elasticities for other combinations of exogenous and endogenous variables should they be required.

The computer program contains a further degree of flexibility in that provision is made to add extra variables and equations to the basic model described by equations (1) to (8). This is achieved by appending a number of rows (equations) and columns (variables) to the final matrix equation. The procedure is simplest if the new equation can be used as it is without first having to eliminate any of the variables that are always endogenous.

1. The matrix to be inverted consists of submatrices with strong diagonal elements. These facilitate inversion. Thus exotic matrix inversion packages are not necessary.

Derivation of the Condensed Structural Equations of the ORANI Module
in Matrix Form

Equation 1 : Price of Capital Goods

Equation (5.4) from paper O-01 can be written

$$\pi_i = \sum_{j=1}^g a_{j1i}^{(2)} p_{j1} + \sum_{j=1}^g a_{j2i}^{(2)} p_{j2} \quad (5.4)$$

where i takes values 1 to g .

π_i is (the proportional change in) the price of capital goods of type i ($i=1$ to g).

p_{j1} and p_{j2} are the prices of domestic and imported goods of type j ($j=1$ to g).

$a_{jsi}^{(2)}$ is the share in the total cost of creating one unit of capital for industry i which is accounted for by inputs of good j from source s .

This is a differential form of the statement that the price of a unit of capital good specific to a particular industry is equal to the sum of the costs of the domestic and imported goods used in its creation. It assumes that no primary factor or other costs are involved in the creation of capital. The use of primary factors in the creation of capital is accounted for via inputs from the construction industries.

Equation (5.4) can be written in matrix form as

$$\pi = A_1^{(2)} P_1 + A_2^{(2)} P_2 \quad (1)$$

where $A_1^{(2)}$ and $A_2^{(2)}$ are the transposes of the matrices $a_{j1i}^{(2)}$ and $a_{j2i}^{(2)}$.

The last two matrices consist of shares derived from the investment columns of the input output table. The flows are converted to shares of total investment by dividing each element by its column total (the sum over both domestic and imported inputs).

1. Throughout appendix 1, Dixon's paper on "The Theoretical Structure of the ORANI Module" (op. cit.) will be referred to as paper O-01.

Equation 2 : Domestic Commodity Prices

Equation (5.3) of paper O-01 can be written

$$P_{i1} = \sum_{j=1}^g a_{j1i}^{(1)} P_{j1} + \sum_{j=1}^g a_{j2i}^{(1)} P_{j2} + a_{(g+1)1i}^{(1)} P_{(g+1)1i} \\ + a_{(g+1)2i}^{(1)} q_{2i} + a_{(g+1)3i}^{(1)} P_{(g+1)3i} + a_{(g+2).i}^{(1)} q_{4i}$$

where $P_{(g+1)1i}$ is the cost of a unit of labour in industry i ;
 ie. the average (annual) wage in industry i ;

$q_{2i} \equiv P_{(g+1)2i}$ is the annual rental value of a unit of
 fixed capital used in industry i ;

$P_{(g+1)3i}$ is the annual rental value of a unit of land
 used in industry i ;

$q_{4i} \equiv P_{(g+2).i}$ is the annual cost to industry i of a unit
 of "other costs of production" such as taxes,
 subsidies, inventories and costs of holding
 liquidity ;

$a_{jsi}^{(1)}$ is the share in the total cost of producing
 good i which is accounted for by inputs of
 type j and source s .

This equation is the differential form of the statement that the domestic price of a domestically produced good equals the sum of the costs involved in its production. The first two terms refer to the inputs of domestically produced and imported goods, the next three refer to the inputs of primary factors (wages, fixed capital and land), and the last allows for other costs of production.

The model allows the rental value of capital and the value of "other costs" to vary independently from industry to industry. For the remainder of this paper these two variables are represented by q_{2j} and q_{4j} respectively.

In paper O-01 it was assumed that there was only one type of labour and that the wage rate was the same for all industries (see equation (6.3) in appendix 4 of paper O-01). This equation is now replaced by

$$P_{(g+1)1j} = \sum_{\ell=1}^h S_{(g+1)1(\ell)j} q_{1\ell} \quad (\text{new 6.3})$$

where $S_{(g+1)1(\ell)j}$ is the share of the total wage bill for industry j that is accounted for by occupation ℓ ;

$q_{1\ell} = P_{(g+1)1(\ell)} = P_{(g+1)1(\ell)j}$ is the wage rate for occupation ℓ which is assumed to be the same for all industries .

The new equation (6.3) is a differential form of the identity

$$P_{(g+1)1j} X_{(g+1)1j}^{(1)} = \sum_{\ell=1}^h P_{(g+1)1(\ell)j} X_{(g+1)1(\ell)j}^{(1)} \quad \text{which states}$$

that the total wage bill for an industry is the sum of the wage bills in that industry for the different occupations.

In a similar manner equation (6.5) is replaced by

$$P_{(g+1)3j} = \sum_{r=1}^n S_{(g+1)3(r)j} q_{3r} \quad (\text{new 6.5})$$

where $S_{(g+1)3(r)j}$ is the share of land rental costs in industry j that is accounted for by land of type r ;

$q_{3r} = P_{(g+1)3(r)} = P_{(g+1)3(r)j}$ is the rental value of land of type r which is assumed to be the same for all industries .

Equation (5.3) can now be written

$$p_{i1} = \sum_{j=1}^g a_{j1i}^{(1)} p_{j1} + \sum_{j=1}^g a_{j2i}^{(1)} p_{j2} + \sum_{\ell=1}^h a_{(g+1)1(\ell)i}^{(1)} q_{1\ell} \\ + a_{(g+1)2i}^{(1)} q_{2i} + \sum_{r=1}^n a_{(g+1)3(r)i}^{(1)} q_{3r} + a_{(g+2).i}^{(1)} q_{4i}$$

where $a_{(g+1)1(\ell)i}^{(1)} = a_{(g+1)li}^{(1)} S_{(g+1)1(\ell)i}$

and $a_{(g+1)3(r)i}^{(1)} = a_{(g+1)3i}^{(1)} S_{(g+1)3(r)i}$.

This means that $a_{(g+1)1(\ell)i}^{(1)}$ and $a_{(g+1)3(r)i}^{(1)}$ are the shares of the total cost of producing good i that are accounted for by wages in occupation ℓ and the rental of land of type r .

Equation (5.3) can now be written in matrix form

$$p_1 = A_1^{(1)} p_1 + A_2^{(1)} p_2 + A_3^{(1)} q_1 + A_4^{(1)} q_2 + A_5^{(1)} q_3 + A_6^{(1)} q_4 \quad (2)$$

The 6 matrices $A_1^{(1)}$ to $A_6^{(1)}$ are directly related to the 6 matrices in the Intermediate Usage columns of the input output table. The flows in the input output table are first converted to shares of total production costs by dividing each element by its column total (the sum over all goods, primary factors and other costs). $A_1^{(1)}$ to $A_6^{(1)}$ are then the transposes of these six total cost share matrices, except that $A_4^{(1)}$ and $A_6^{(1)}$ need to be converted from vectors to diagonal matrices.

Equation 3 : Investment

The investment equations from paper O-01 can be written

$$-\beta_i (k_i - n_{2i}) + Q_i (q_{2i} - \pi_i) = \lambda \quad (4.8)$$

$$k_i = (1-g_i)n_{2i} + g_i i_i \quad (4.9)$$

$$\sum_{j=1}^g z_j (\pi_j + i_j) = \theta \quad (4.10)$$

The notation is considerably different from that used in paper O-01.

$n_{2i} \equiv k_i(t-1)$ is the capital stock available for use throughout the current year .

$k_i \equiv k_i(t)$ is the capital stock available for use throughout the next year .

$i_i \equiv y_i$ is capital investment made in the current year .

$\theta \equiv i$ is the aggregate value of investment .

λ is the anticipated economy wide MEC for the following year, where MEC is the expected rate of return on an additional unit of capital .

z_i is the share in the total cost of investment accounted for by investment in industry i . (These costs involve summation over both domestic and imported inputs).

β_i is the elasticity of MEC with respect to the level of capital stock .

Q_i is the ratio of gross MEC (without allowance for depreciation) to net MEC .

g_i is a measure of the growth in the capital stock of industry i .

Equation (4.8) is the fundamental investment equation. It is the condition that after installation of new capital in the current year the anticipated MEC in the following year will be the same for all industries. It is assumed that the decision to invest in new capital is made at the beginning of the current year, but the new capital is not installed until the end of the current year. Equation (4.9) relates the change in capital stock over one year to investment and depreciation. Equation (4.10) is the budget constraint on total investment.

k_i can be eliminated from equation (4.8) by means of equation (4.9), giving

$$\beta_i g_i (\eta_{2i} - i_i) + Q_i (q_{2i} - \pi_i) = \lambda \quad (A)$$

An expression for λ is obtained by writing equation (A) as an expression for i_i and then substituting this into equation (4.10).

$$\lambda = \frac{1}{\sum_j \left(\frac{z_j}{\beta_j g_j} \right)} \left(\sum_j z_j \pi_j + \sum_j z_j \eta_{2j} + \sum_j \frac{Q_j z_j}{\beta_j g_j} (q_{2j} - \pi_j) - \theta \right)$$

Next substitute this expression for λ into equation A and put $b_i = \frac{1}{\beta_i g_i}$

$$\text{and } \rho = \sum_j b_j z_j = \sum_j \frac{z_j}{\beta_j g_j}$$

After rearrangement of terms this gives

$$i_i = (\eta_{2i} - \frac{b_i}{\rho} \sum_j z_j \eta_{2j}) + b_i Q_i (q_{2i} - \pi_i) - \frac{b_i}{\rho} \sum_j Q_j b_j z_j (q_{2j} - \pi_j) - \frac{b_i}{\rho} \sum_j z_j \pi_j + \frac{b_i}{\rho} \theta$$

This can be written in vector form as

$$i = G_2 \eta_2 + G_3 q_2 + G_4 \pi + G_5 \theta \quad (3)$$

where G_2 , G_3 and G_4 are matrices of dimension g by g with elements in the i^{th} row and j^{th} column given by

$$G_{2ij} = I_{ij} - \frac{b_i z_j}{\rho},$$

$$G_{3ij} = b_i Q_i I_{ij} - \frac{b_i}{\rho} Q_j b_j z_j = b_j Q_j G_{2ij},$$

$$G_{4ij} = -G_{3ij} - \frac{b_i z_j}{\rho} = -G_{3ij} + G_{2ij} - I_{ij},$$

and G_5 is a vector of length g with elements $\frac{b_i}{\rho}$.

β_i , Q_i and g_i are parameters which are estimated outside the model. The shares z_i are determined from the input-output table; b_i and ρ are formed from β_i , g_i and z_i .

N.B. For all elements of the G matrices to be finite it is necessary that all b_i be finite, which requires $\beta_i \neq 0$ and $g_i \neq 0$ for all i .

Equation 4 : Household Consumption

Equation (3.17) from paper 0-01 can be written

$$x_{is}^{(3)} = \sum_{j=1}^g \eta_{isj1} P_{j1} + \sum_{j=1}^g \eta_{isj2} P_{j2} + \epsilon_{is} C \quad (3.17)$$

This is a differential form of the equation

$$X_{is}^{(3)} = F_{il}(P_{11} \dots P_{g1}; P_{12} \dots P_{g2}; C) \text{ which states that } X_{is}^{(3)}$$

($\equiv C_{is}$ in paper 0-01), the household consumption (physical quantity) of good i from source s , is a function of domestic prices (P_{j1}), import prices (P_{j2}) and total consumption expenditure (C). η_{isjt} is the elasticity of demand for good i from source s with respect to price P_{jt} , and ϵ_{is} is the expenditure elasticity of demand for good i from source s , i.e., elasticity of demand with respect to total household expenditure.

Equations (3.14), (3.15) and (3.16) of paper 0-01 relate the elasticities η_{isjt} , which distinguish between domestic ($s=1$) and imported ($s=2$) sources of goods and between domestic ($t=1$) and import ($t=2$) prices, to the "outside elasticities" η_{ij} where no such distinctions are made. The relationships can be summarized as

$$\eta_{isjt} = (\eta_{ij} + I_{ij} \sigma_i^{(3)}) S_{jt}^{(3)} - I_{ij} I_{st} \sigma_i^{(3)},$$

where

$S_{jt}^{(3)}$ is the share of the total expenditure on good j which is accounted for by purchases from source t ;

$\sigma_i^{(3)}$ is the elasticity of substitution between alternative sources for household consumption of good i ;

η_{ij} is the elasticity of demand for good i (from all sources) with respect to the price of good j (a weighted average of domestic and import prices) .

The expression for η_{isjt} is derived in paper O-01 by maximizing a generalized utility function subject to the budget constraint on aggregate household expenditure. It is assumed that substitution between imported and domestic goods can be described by a CES function.

The model at present is concerned only with household consumption of domestically produced goods, $s=1$. Equation (3.17) can now be written in matrix form as

$$x_1^{(3)} = \eta_{11} p_1 + \eta_{12} p_2 + \epsilon c \quad (4)$$

where $\eta_{11} = (\eta + \hat{\sigma}^{(3)}) \hat{S}_1 - \hat{\sigma}^{(3)}$

and $\eta_{12} = (\eta + \hat{\sigma}^{(3)}) \hat{S}_2$.

For those industries where household consumption of both imports and domestic goods is zero, the domestic shares S_1 are set equal to 1.0.

$\eta_{ij}, \sigma_i^{(3)}$ and ϵ_i are parameters which are estimated outside the model. As explained in paper O-01, $\epsilon_{i1} = \epsilon_{i2} = \epsilon_i$. The shares $S_{jt}^{(3)}$ are derived from the input output table.

Equation 5, 6, 7 : Factor Demands

Equations (6.7), (6.8) and (6.9) of paper 0-01 are differential forms of the equations which describe the balance between supply and demand for each of labour, capital and land. In paper 0-01 it was assumed that there was only one type of labour. Equation (6.7) is now replaced by

$$n_{1\ell} = \sum_{j=1}^g L_{\ell j} x_{(g+1)1(\ell)j} \quad (\text{new 6.7})$$

which allows for h different types of labour ($\ell = 1$ to h), with demand equalling supply in each occupation.

$n_{1\ell}$ is the supply of labour in occupation ℓ
 $x_{(g+1)1(\ell)j}$ is the demand in industry j for labour of occupation ℓ
 $L_{\ell j}$ is the share of all persons employed in occupation ℓ which work in industry j.
 Hence $\sum_{j=1}^g L_{\ell j} = 1$.

Equation(6.7) is the differential form of the balance equation for labour in each occupation, $n_{1\ell} = \sum_{j=1}^g X_{(g+1)1(\ell)j}$, where n and X are measured in numbers of persons.

It is assumed that substitution between inputs of the three primary factors can be described by CES production functions. However within the labour category it is also assumed that there is no substitution between different occupations. The production technology for a given industry requires fixed ratios between the numbers of workers in each occupation. If the number of persons in occupation ℓ of industry j increases by one percent then the number of persons in occupation m of industry j also increases by one percent.

This is expressed in the form of a new equation

$$x_{(g+1)1(\ell)j} = x_{(g+1)1j} \quad (3.6(iii))$$

Similarly there is no substitution between types of land. This results in another new equation

$$x_{(g+1)3(r)j} = x_{(g+1)3j} \quad (3.6(iv))$$

In the case of capital stock the question of substitution between different types of capital does not arise because it is assumed that there is one type of capital unique to each industry.

The criterion of cost minimization in production, together with a CES production function, leads to the demand equations for primary factor inputs

$$x_{(g+1)sj} = x_j - \alpha_j \left(p_{(g+1)sj} - \sum_{s=1}^3 p_{(g+1)sj} S_{(g+1)sj}^{(1)} \right) \quad (3.6(ii))$$

where x_j is the total output of industry j ;
 $\alpha_j \equiv \sigma_{(g+1)j}^{(1)}$ is the elasticity of substitution between primary factors ;
 $S_{(g+1)sj}^{(1)}$ is the share of the total primary factor costs for industry j which is accounted for by primary factor s .

The primary factors are labour ($s=1$), capital stock ($s=2$) and land ($s=3$).

Using equations (3.6(ii)) and (3.6(iii)) to replace $x_{(g+1)1(\ell)j}$ in equation (6.7) gives

$$n_{1\ell} = \sum_{j=1}^g L_{\ell j} \left[x_j - \alpha_j \left(p_{(g+1)1j} - \sum_{s=1}^3 p_{(g+1)sj} S_{(g+1)sj}^{(1)} \right) \right] .$$

Next use the new equations (6.3) and (6.5) (wage rate in occupation ℓ is the same for all industries, and rental value of land type r is the same for all industries) to obtain

$$n_{1\ell} = \sum_j L_{\ell j} \left[x_j - \alpha_j \left((1 - S_{(g+1)1j}^{(1)}) \left(\sum_{\ell=1}^h S_{(g+1)1(\ell)} q_{1\ell} \right) - S_{(g+1)2j}^{(1)} q_{2j} - S_{(g+1)3j}^{(1)} \left(\sum_{r=1}^n S_{(g+1)3(r)j} q_{3r} \right) \right) \right] .$$

The factor demand equations for capital stock and land can be derived in a similar manner starting from

$$n_{2j} = x_{(g+1)2j} \quad (6.8)$$

and a revised version of equation (6.9) which now allows for n different types of land.

$$n_{3r} = \sum_{j=1}^g R_{rj} x_{(g+1)3(r)j} \quad (\text{new 6.9})$$

where n_{2j} is the supply of capital stock in industry j ;
 n_{3r} is the supply of land of type r ;
 $x_{(g+1)2j}$ is the demand for capital of type j ;
 $x_{(g+1)3(r)j}$ is the demand for land of type r by industry j ;
 R_{rj} is the share of the land of type r which is used in industry j ie. $\sum_{j=1}^g R_{rj} = 1$.

The factor demand equations are derived from equations (6.8) and (6.9) by making use of equations (3.6(ii)), (3.6(iv)), (6.3) and (6.5).

In matrix form the three factor demand equations are

$$\begin{aligned} \eta_1 &= Lx + L\hat{\alpha}(\hat{S}_1 - I)Hq_1 + L\hat{\alpha}\hat{S}_2q_2 + L\hat{\alpha}\hat{S}_3Nq_3 \\ \eta_2 &= Ix + \hat{\alpha}\hat{S}_1Hq_1 + \hat{\alpha}(\hat{S}_2 - I)q_2 + \hat{\alpha}\hat{S}_3Nq_3 \\ \eta_3 &= Rx + R\hat{\alpha}\hat{S}_1Hq_1 + R\hat{\alpha}\hat{S}_2q_2 + R\hat{\alpha}(\hat{S}_3 - I)Nq_3, \end{aligned}$$

or

$$\eta_1 = L_0x + L_1q_1 + L_2q_2 + L_3q_3 \quad (5)$$

$$\eta_2 = K_0x + K_1q_1 + K_2q_2 + K_3q_3 \quad (6)$$

$$\eta_3 = N_0x + N_1q_1 + N_2q_2 + N_3q_3 \quad (7)$$

where L is a matrix of dimensions $h \times g$. Element L_{lj} is the share of all persons employed in occupation l which work in industry j .

$$\text{Hence } \sum_j L_{lj} = 1.$$

R_{rj} (dimensions $n \times g$) is the share of all land of type r which is used in industry j , i.e. $\sum_j R_{rj} = 1$.

H_{il} (dimensions $g \times h$) is the share of total wages in industry i that is accounted for by occupation l , i.e., $\sum_l H_{il} = 1$.

N_{ir} (dimensions $g \times n$) is the share of total land rental costs in industry i that is accounted for by land of type r , i.e., $\sum_r N_{ir} = 1$.

S_{1j} , S_{2j} , S_{3j} are the shares of total primary factor costs in industry j that are accounted for by total wages, rental of capital, and total costs of renting land respectively. They are obtained from the input-output table.

α_j is a vector estimated outside the model.

Equation 8 : Market Clearing

Equation (6.6) from paper O-01 can be written

Total domestic output

$$\begin{aligned}
 x_i = & \sum_{j=1}^g x_{ilj}^{(1)} B_{ilj}^{(1)} \quad \text{intermediate usage} \\
 & + \sum_{j=1}^g x_{ilj}^{(2)} B_{ilj}^{(2)} \quad \text{usage in capital formation} \\
 & + x_{il}^{(3)} B_{il}^{(3)} \quad \text{household consumption} \\
 & + x_{il}^{(4)} B_{il}^{(4)} \quad \text{exports} \\
 & + x_{il}^{(5)} B_{il}^{(5)} \quad \text{other uses} .
 \end{aligned}$$

This equation is a differential form of the statement that for each domestically produced good the supply (or total domestic output) equals the demand (or sum of the uses). The B's are the shares of the total domestic output going to the various users. $x_{il}^{(3)}$, $x_{il}^{(4)}$ and $x_{il}^{(5)}$ are equivalent to c_{il} , e_{il} and f_{il} as used in paper O-01.

The criterion of cost minimization in domestic production, subject to CES substitution between inputs of imported and domestic goods, leads to demand equations for the input of domestically produced goods.

$$x_{ilj}^{(1)} = x_j^{(1)} - \sigma_{ij}^{(1)} \{p_{i1} \left[1 - S_{ilj}^{(1)} \right] - p_{i2} S_{i2j}^{(1)}\} \quad (3.6(i)) \text{ for } s=1$$

Similarly, cost minimization in the formation of capital goods, under the constraint of CES substitution between inputs of imported and domestic goods, leads to demand equations for the input of domestically produced goods.

$$x_{ilj}^{(2)} = x_j^{(2)} - \sigma_{ij}^{(2)} \{p_{i1} (1 - S_{ilj}^{(2)}) - p_{i2} S_{i2j}^{(2)}\} \quad (3.10) \text{ for } s=1 .$$

$\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ are the elasticity of substitution between domestic and imported inputs of good i for production of good j , and the formation of capital of type j , respectively.

$x_{ilj}^{(1)}$ and $x_{ilj}^{(2)}$ ($\equiv y_{ilj}$ in paper O-01) are the demands for domestic goods i in the production of good j and the formation of capital of type j .

$x_j^{(1)}$ ($\equiv x_j$) is the total domestic output of good j , which is also equal to the total usage of domestic good j .

$x_j^{(2)}$ ($\equiv i_j \equiv y_j$) is the total capital created in industry j , i.e., the total (annual) capital investment in industry j .

$S_{isj}^{(1)}$ is the share of the total cost of good i in the production of good j that is accounted for by source s .

$$\text{Hence, } S_{ilj}^{(1)} + S_{i2j}^{(1)} = 1 .$$

$$\text{Similarly, for capital formation, } S_{ilj}^{(2)} + S_{i2j}^{(2)} = 1 .$$

Hence, (3.6(i)) and (3.10) can be written

$$x_{ilj}^{(1)} = x_j - \sigma_{ij}^{(1)} S_{i2j}^{(1)} (p_{i1} - p_{i2})$$

$$x_{ilj}^{(2)} = i_j - \sigma_{ij}^{(2)} S_{i2j}^{(2)} (p_{i1} - p_{i2}) .$$

Substitution of these expressions for $x_{ilj}^{(1)}$ and $x_{ilj}^{(2)}$ into equation (6.6) gives

$$\begin{aligned} x_i = & \sum_{j=1}^g B_{ilj}^{(1)} x_j + \sum_{j=1}^g B_{ilj}^{(2)} i_j + B_{il}^{(3)} x_{i1}^{(3)} + B_{il}^{(4)} x_{i1}^{(4)} + B_{il}^{(5)} x_{i1}^{(5)} \\ & - \sum_{j=1}^g \{ \sigma_{ij}^{(1)} S_{i2j}^{(1)} B_{ilj}^{(1)} + \sigma_{ij}^{(2)} S_{i2j}^{(2)} B_{ilj}^{(2)} \} (p_{i1} - p_{i2}) . \end{aligned}$$

In matrix notation

$$x = \beta^{(1)} x + \beta^{(2)} i + \beta^{(3)} x_1^{(3)} + \beta^{(4)} x_1^{(4)} + \beta^{(5)} x_1^{(5)} - \psi p_1 + \psi p_2 \quad (8)$$

where

$$\beta^{(1)} \equiv B_1^{(1)}, \beta^{(2)} \equiv B_1^{(2)}, \beta^{(3)} \equiv \hat{B}_1^{(3)}, \beta^{(4)} \equiv \hat{B}_1^{(4)}, \beta^{(5)} \equiv \hat{B}_1^{(5)}$$

are matrices of dimension $g \times g$ ($\beta^{(3)}$, $\beta^{(4)}$ and $\beta^{(5)}$ are diagonal) .

ψ is a diagonal matrix the elements of which are obtained by

term by term multiplication of corresponding terms from three matrices, followed by row summation.

Referring to the input-output table, $B_1^{(1)}$ to $B_1^{(5)}$ are the flows in the first g rows expressed as shares of the total outputs for these rows (the sums over all users). The shares $S_{i2j}^{(1)}$ and $S_{i2j}^{(2)}$ are obtained from the input-output table.

$\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ are parameters estimated outside the model.