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The Use of Adjustment Cost Investment Models in Intertemporal Computable General Equilibrium Models

by

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ABSTRACT

Wilcoxen (1989) introduced a paradigm whereby the shadow price of capital that characterises the solution to an intertemporal cost of adjustment investment model is used to provide the explicit dynamic linkages for sequences of computable general equilibrium models. This paper investigates the possible specification of adjustment cost investment models which are considered useful for such an approach, and considers possible approaches to the estimation of the parameters of such models.

THE USE OF ADJUSTMENT COST INVESTMENT MODELS IN INTERTEMPORAL COMPUTABLE GENERAL EQUILIBRIUM MODELS

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1. Introduction

Two possible criticisms of the ORANI model (Dixon et.al (1982)) are that

(a) the model is not explicitly dynamic

and

(b) in long-run closures, output responses to price changes tend to be quite large.

In Horridge, Powell and Wilcoxen (1990) and Wilcoxen (1990), the possibility is raised that these criticisms are related, and that an appropriate response may be to explicitly model the adjustment of capital stock within a framework of costs of adjustment that are non-linearly related to the rapidity of adjustment. A prototype example of such a computable general equilibrium model is given in Wilcoxen (1989). A general overview of the integration of cost of adjustment models into CGE models is provided in Section 2. Section 3 then considers the appropriate specification of technology to allow such a general integration, and Section 4 turns to the econometric estimation of the parameters of this technology.

2. An Overview of the Integration of Adjustment Cost Investment Models into CGE Models.

The paradigm of the adjustment cost investment models is that the interests of the owners of the firm are best served if the managers of the firm take decisions based on the maximization of the value of the firm defined as the present value of the future flow of (after-tax) dividends $\int_{\tau}^{\infty} e^{-r(t-\tau)} D(t)\,dt$. The technology of the firm, and the institutional environment in which it operates, are characterised by the functional form of D(t). In general, we can write

$$D(t) = D(K(t),L(t),I(t),P(t),T(t);\theta) \tag{2.1}$$

where each of the arguments of D are vectors of various classes of arguments. (In the general discussion, the vector nature of these arguments will typically be ignored.)

K(t) and I(t) refer to the levels of, and gross investment in, respectively, quasi-fixed factors ("capital"). These are typically related by a transition equation such as

$$\dot{K}(t) = I(t) - \delta K(t) \;. \tag{2.2}$$

The model will be within the class of adjustment cost models provided that D( ) is non-linear (in fact concave) in I. (For an elaboration of this point, see McLaren (1990).)

L(t) refers to the flow of services of variable factors ("labour"), where variable factors are distinguished by the fact that their rates of change do not enter into the specification of D( ). The paths of I(t) and L(t) are to be chosen by the firm, and hence are control variables. The path of I(t) determines the path of K(t), via (2.2), and K(t) is hence a state variable. If the firm produces a single output Y, its supply could be characterised by a production function such as

$$Y(t) = f(K(t),L(t),I(t)) \tag{2.3}$$

which is implicit in the form of D( ). Multiple output production is not easily handled in this representation, and will be considered below where it is more easily accommodated.

The environment within which the firm operates is captured by P and T. Typical of the elements of P are the prices of outputs and inputs, Y, I and L, that are taken as given by the firm when choosing I(t) and L(t), but at a later stage will be determined by the interaction of the demands and supplies of this model with a CGE model at each point in time. Typical of the elements of T are institutional factors such as tax rates and tax rules for the treatment of depreciation, investment, etc. that are more in the nature of policy variables.

Finally, θ is a vector of parameters that, along with a specification of functional form, completely determines D( ).

Given time paths of P(t) and T(t), $\tau \leq t \leq \infty$, a discount rate r, and an initial stock of capital K_τ, at time τ the firm chooses time paths of I(t) and L(t) to maximize

$$\int_\tau^\infty e^{-r(t-\tau)} D(K,L,I,P,T;\theta)\, dt \qquad (2.4)$$

subject to

$$\dot{K} = I - \delta K$$

and $$K(\tau) = K_\tau \text{ given.}$$

First-order conditions for this problem may be stated in terms of the (present value) Hamiltonian $\tilde{H}$ and (present value) multiplier $\Lambda(t)$. Define

$$\tilde{H} = e^{-r(t-\tau)} D + \Lambda(I-\delta K) . \qquad (2.5)$$

Then the control variables I and L should be chosen to maximize $\tilde{H}$ at each point in time, so that, assuming interior solutions, necessary conditions are[1]:

$$\tilde{H}_I = 0 \qquad \text{i.e.,} \qquad e^{-r(t-\tau)} D_I + \Lambda = 0 \tag{2.6a}$$

$$\tilde{H}_L = 0 \qquad \text{i.e.,} \qquad e^{-r(t-\tau)} D_L = 0 \tag{2.6b}$$

Further, the optimal paths satisfy the transition equations:

$$\dot{K} = \tilde{H}_\Lambda \qquad \text{i.e.,} \qquad \dot{K} = I - \delta K \tag{2.6c}$$

$$\dot{\Lambda} = -\tilde{H}_K \qquad \text{i.e.,} \qquad \dot{\Lambda} = -e^{-r(t-\tau)} D_K + \delta\Lambda \tag{2.6d}$$

and the endpoint conditions

$$K(\tau) = K_\tau \tag{2.6e}$$

$$\lim_{t\to\infty} \Lambda(t)K(t) = 0 \,. \tag{2.6f}$$

In models such as these, it is convenient to define the (current value) Hamiltonian H and the (current value) shadow price λ:

$$\lambda = \Lambda\, e^{r(t-\tau)}$$

$$H = \tilde{H}\, e^{r(t-\tau)}$$

$$= D + \lambda(I-\delta K)$$

Then $\dot{\lambda} = re^{r(t-\tau)}\Lambda + \dot{\Lambda}e^{r(t-\tau)}$, the control variables I and L maximize H at each point in time, and the necessary conditions (2.6) become:

$$H_I = 0 \qquad \text{i.e.,} \qquad D_I + \lambda = 0 \tag{2.7a}$$

$$H_L = 0 \qquad \text{i.e.,} \qquad D_L = 0 \tag{2.7b}$$

$$\dot{K} = I - \delta K \tag{2.7c}$$

$$\dot{\lambda} = (r+\delta)\lambda - D_K \tag{2.7d}$$

$$K(\tau) = K_\tau \tag{2.7e}$$

$$\lim_{t\to\infty} \lambda(t)\, e^{-r(t-\tau)}\, K(t) = 0\ . \tag{2.8f}$$

The variables represented by P and T are taken as given by the firm, and in general will represent expectations of the future paths of P and T. Consider first the case of static expectations, $P(t) = \bar{P}$, $t \geq \tau$, $T(t) = \bar{T}$, $t \geq \tau$, and for simplicity, absorb r into $\bar{P}$. Now for any functional form of $D(\cdot)$ it is useful to characterise two alternative representations of solutions to system (2.7).

(a) <u>Open Loop Solutions</u>

There exist functions of time

$$K(t) = K^o(t;K_\tau,\bar{P},\bar{T},\theta) \tag{2.8a}$$

$$\lambda(t) = \lambda^o(t;K_\tau,\bar{P},\bar{T},\theta) \tag{2.8b}$$

$$I(t) = I^o(t;K_\tau,\bar{P},\bar{T},\theta) \tag{2.8c}$$

$$L(t) = L^o(t;K_\tau,\bar{P},\bar{T},\theta) \tag{2.8d}$$

which satisfy system (2.7). In principle, such numerical solutions can always be constructed to satisfy (2.7), given a specification of D( ), and for certain specifications of D( ) analytical solutions would be available.

(b) <u>Closed Loop Solutions</u>

There exist functions of the state variable K and exogenous variables $\bar{P}$, $\bar{T}$:

$$\lambda = \lambda^c(K;\bar{P},\bar{T},\theta) \tag{2.9a}$$

$$I = I^c(K;\bar{P},\bar{T},\theta) \tag{2.9b}$$

$$L = L^c(K;\bar{P},\bar{T},\theta) \tag{2.9c}$$

such that the path of $K(t)$ generated as a solution of $\dot{K} = I^c - \delta K$, $K(\tau) = K_\tau$, is $K^o(t;K_\tau,\bar{P},\bar{T},\theta)$, and satisfies

$$\lambda^o \equiv \lambda^c(K^o(t;K_\tau,\bar{P},\bar{T},\theta);\bar{P},\bar{T},\theta)$$

etc.

A closely related closed loop solution is the value function

$$V(K,\bar{P},\bar{T};\theta)$$

where V evaluated at K_τ is defined as the maximized value of (2.4) and a basic result from optimal control theory is the relationship:

$$V_K = \lambda .$$

This relationship reinforces the interpretation of the current value costate variable λ as the shadow price of a unit of capital, the marginal contribution, at time t, of an extra unit of capital at time t, when optimally utilized. In general, there will be a close relationship between the functional form of D( ) and the functional form of V( ). The value function V is analogous to the profit function of static theory, and a rule allowing the derivation of the functional forms of I^c and L^c from the functional form of V would be an intertemporal analogue of Hotelling's Lemma. Duality between V and D, and the form of this analogue to Hotelling's Lemma, are explored in Cooper and McLaren (1980, 1987a) and Epstein (1981).

From the standpoint of econometric estimation of the parameters θ, the closed loop solutions (2.9) are usually considered superior - they specify the investment demands and labour demands (and implicitly output supplies) at each point in time as functions of the existing level of capital and the exogenously given variables $\bar{P}$ and $\bar{T}$. The difficulty with this approach is the derivation of the functional forms of $I^c(\ )$ and $L^c(\ )$ consistent with a specification of an appropriate functional form of $D(\ )$. To this end, the dual specification of the value function V is potentially useful.

From the standpoint of CGE modelling, the choice between solution methods hinges on different criteria. If analytic closed loop solutions of the form (2.9) are available, an appropriate way to integrate an intertemporal model of investment into a CGE model might be to use N grid-points to write

$$K(t) = K_\tau \quad \text{given} \tag{2.10a}$$

$$[K(t+h) - K(t)] \,/\, h = I(t) - \delta K(t) \tag{2.10b}$$

$$I(t) = I^c(K(t),\bar{P},\bar{T},\theta) \tag{2.10c}$$

$$L(t) = L^c(K(t),\bar{P},\bar{T},\theta) \tag{2.10d}$$

for $t = \tau,\ \tau+h,\ \ldots\ \tau+\left(\frac{N}{h}-1\right).h$

(plus appropriate output supply equations) and appending, for each t, the other market clearing equations from a CGE to give an intertemporal CGE model, with solutions at each point in time linked together by the state variable (vector) K(t). The assumption of static expectations would be consistent with replacing, at each t, $\bar{P}$ by P(t), the market clearing vector, and $\bar{T}$ by T(t). An obvious extension at this stage would be to add the value and dividend functions, V and D, as inputs to a life-cycle model of consumption to strengthen the intertemporal linkages of the model.

The open loop solutions, (2.8), if available in analytical form, could similarly be added, at each t, to a CGE model. (If only numerical solutions are available, solution would be burdensome.) In Wilcoxen (1989) an intermediate approach is taken. First, it is recognized that variable factors (L) can be optimized out at each point in time (for an elaboration on this idea, see Cooper and McLaren (1987b), and also see below). Second, the role of the co-state variable as a shadow price is recognized (at each point in time λ is a "sufficient statistic" for I), and a partial solution to system (2.7) is written as

$$K(t) = K_\tau \qquad t = \tau \tag{2.11a}$$

$$[K(t+h) - K(t)]/h = I(t) - \delta K(t) \tag{2.11b}$$

$$I(t) = I^F(\lambda(t),K(t),\bar{P},\bar{T},\theta) \tag{2.11c}$$

$$[\lambda(t+h) - \lambda(t)]/h = (r+\delta)\lambda(t) - D_K(K(t),L(t),I(t),\bar{P},\bar{T},\theta) \tag{2.11d}$$

$$L(t) = L^c(K(t),\bar{P},\bar{T},\theta) \tag{2.11e}$$

$$\text{for } t = \tau,\ \tau+h,\ \ldots\ \tau+\left(\frac{N}{h}-1\right)h \tag{2.11f}$$

$$\lambda(\tau+N) = \lambda^{ss} \tag{2.11g}$$

to be solved together with the remainder of the CGE model specified for each t, where λ^{ss} is the steady state solution for λ. Thus the numerical solution of the necessary conditions (2.7) is integrated with the solution of the CGE model, in much the same way as such models solve for marked clearing prices. The superscript F on I stands for Frisch, to indicate that this equation for investment demand is "shadow price of capital" conditioned.

In practice, it is desirable to allow the decision-maker to hold non-static expectations about the exogenous variables P and T. In general, let $\{P_\tau\}$ stand for the function P(t), $\tau \le t \le \infty$, and define $\{T_\tau\}$ similarly. Then $V(K_\tau,\bar{P},\bar{T},\theta)$, a function, is replaced by the functional

$$V(K_\tau,\{P_\tau\},\{T_\tau\},\theta)$$

and the closed loop solutions λ^C and I^C are similarly functionals (but not L^C). However, the role of λ as a sufficient statistic is unaffected, and system (2.11) remains valid if $\bar{P}$ is replaced everywhere by P(t) (and, indeed, this includes r being replaced by r(t)), and $\bar{T}$ by T(t). The details of the necessary conditions in the case of time varying P, T and r will be given in Section 4, after a more convenient characterization of the technology has been introduced in Section 3.

This is the essential insight of Wilcoxen: provided the investment demand equations are "shadow price of capital" conditioned, allowing for model consistent expectations rather than static expectations has no effect on the complexity of the model when specified in this form - in the terminology of Cooper, Madan and McLaren (1990), λ is a "natural" variable, and the approach is "Gormanesque". Of course this should not be surprising given that the essence of the maximum principle of optimal control theory is that the co-state variable, λ, captures the effect of current actions on the future, so that the intertemporal optimization problem (2.4) (the choice of paths I(t) and L(t)) can be reduced to a sequence of static optimization problems (at each t, choose $I(t)$ and $L(t)$ to maximize H) conditional on the appropriate shadow price of capital, $\lambda(t)$. But the Wilcoxen methodology neatly integrates a solution method for the evaluation of the co-state variable (the finite difference method) with the solution of a sequence of contemporaneous CGE models.

To make this procedure operational two problems remain: to specify appropriate functional forms for D( ) (or related functions), and to estimate the parameters θ. Section 3 will discuss the first of these problems, and Section 4 will discuss the second.

3. Functional Forms

3.1 Introduction

The example given in Wilcoxen (1989) is essentially of the form:

$$D = E(K,P) - C(I,P)$$

where $E(K,P) = e(P).K$ and $C(I,P)$ is convex in I: $C(I,P) = P_K I + \theta w I^2$.

Solving (2.7a) gives $I = \frac{1}{2w\theta}(\lambda - P_K)$

and (2.7c), (2.7d) become

$$\dot{K} = \frac{1}{2w\theta}(\lambda - P_K) - \delta K \ .$$

$$\dot{\lambda} = (r+\delta)\lambda - e(P)$$

While this specification of D separable in K and I has the mathematical simplification that the transition equation for λ is independent of K, the economic consequence of this is that the level of gross investment is independent of the size of the firm, as measured by K. This is considered to be an unacceptable empirical restriction (generally being at variance with the typical constant returns to scale specification in ORANI), and we will now turn to relaxing this restriction.

With this in mind, it will turn out to be convenient to consider specifications of adjustment costs in terms of net investment, $\dot{K}$. Thus introduce the modified D function,

$$B(K,\dot{K},L,P,T) \equiv D(K,\dot{K}+\delta K,L,P,T)$$

and note that

$$B_{\dot{K}} = D_I$$

$$B_K = D_K + \delta D_I \ .$$

Hence conditions (2.7a-f) are replaced by:

$$B_{\dot{K}} + \lambda = 0 \tag{3.1a}$$

$$B_L = 0 \tag{3.1b}$$

$$I = \dot{K} + \delta K \tag{3.1c}$$

$$\dot{\lambda} = (r+\delta)\lambda - B_K - \delta\lambda$$

i.e., $$\dot{\lambda} = r\lambda - B_K \ . \tag{3.1d}$$

$$K(\tau) = K_\tau \tag{3.1e}$$

$$\lim_{t\to\infty} \lambda(t)e^{-r(t-\tau)}K(t) = 0 \tag{3.1f}$$

To fix ideas, we will first consider some simple examples, and then consider directions in which the preferred example can be generalized while still preserving the simplicity of approach of the previous section. For the purpose of these examples, static expectations will be maintained.

Using simple examples, it is useful to characterize adjustment cost models of investment into three categories:

(a) <u>Diminishing Returns in Production and in (Net) Investment</u>.

This example is typified by the original Eisner and Strotz (1963) model. Let

$$B(K,\dot{K}) \quad = \quad \beta K - \frac{1}{2}\alpha K^2 - \gamma\dot{K}^2 \ .$$

Then the closed loop solution is the flexible accelerator model

$$\dot{K} \quad = \quad \mu(K - K^*)$$

with μ, K^* functions of r, and of β,α,γ (which themselves may be functions of P,T, etc.). Such models are typified by stationary values of K and λ and the response to a shock may be typified by the $(K,\dot{K})$ phase diagram Figure 3.1. If the system is in equilibrium at point A, a shock (say to β) at time τ will lead to a jump to point B, and gradual adjustment to point C with $\dot{K} = 0$, $K = K^*$. This is the classic rationalization of a determinate investment demand equation.[2]

(b) <u>Constant Returns in Production, Diminishing Returns in Net Investment</u>.

Let $B(K,\dot{K}) = \beta K - \gamma\dot{K}^2$. Then the optimal solution is $\dot{K} = \dot{K}^*(\beta,\gamma)$ where $\dot{K}^*$ is independent of K. Such models are typified by linear growth paths of K, stationary levels of investment, and stationary values of λ, and the response to a shock is given in Figure 3.2. If the system is initially in equilibrium on path A, a shock at time τ will lead to a jump to the path B.

A depreciation rule $\dot{K} = I - \delta K$, with gross investment costing s per unit, could be added to either (a) or (b) with no effect on qualitative results. While the example in Wilcoxen (1989) looks superficially like category (b), it is in fact in category (a):

$$B(K,\dot{K}) \quad = \quad (e-\delta P_K)K - \theta w\delta^2 K^2 - 2\theta w\delta K\dot{K} - \theta w\dot{K}^2$$

The only real difference is the addition of the $K\dot{K}$ cross-product term, but there exist stationary values for K and λ, as in (a). Interestingly, this example does share with category (a) the property that V is linear in K.

(c) <u>Constant Returns in Production and Net Investment</u>.

Let $B(K,\dot{K}) = \beta K - \gamma(\dot{K}/K)^2$, and define $k \equiv \dot{K}/K$, the rate of net investment. Then the solution is

$$k \quad = \quad k^*(\beta,\gamma)$$

$$\text{i.e.,} \qquad \dot{K} \quad = \quad k^*.K\ .$$

This model is typified by constant rate of growth paths of K, stationary <u>rates</u> of investment relative to K, and stationary values of λ. The response to a shock is given in Figure 3.3. A shock at time τ will move the firm from path A to B, and a more appropriate phase diagram is the k,K diagram in Figure 3.4.

The prototype model of category (c) is more in the spirit of ORANI specifications, and will now be considered in more detail.

Figure 3.1

($\dot{K}$,K) Phase Diagram: Diminishing Returns in Production and Net Investment

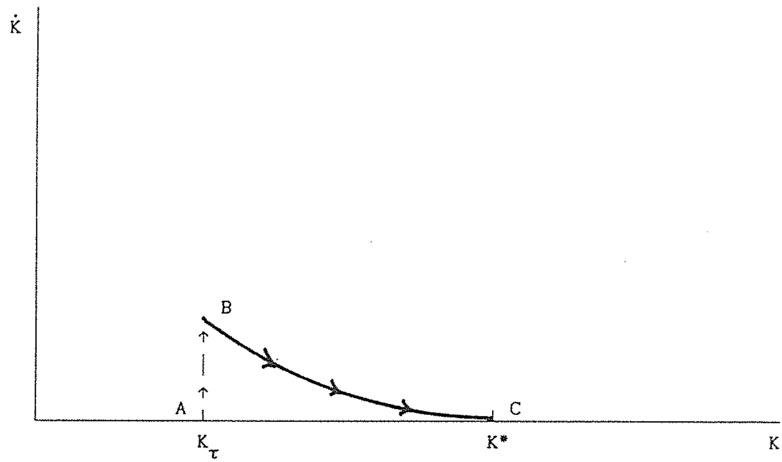


Figure 3.2

($\dot{K}$,K) Phase Diagram: Constant Returns in Production, Diminishing Returns in Net Investment

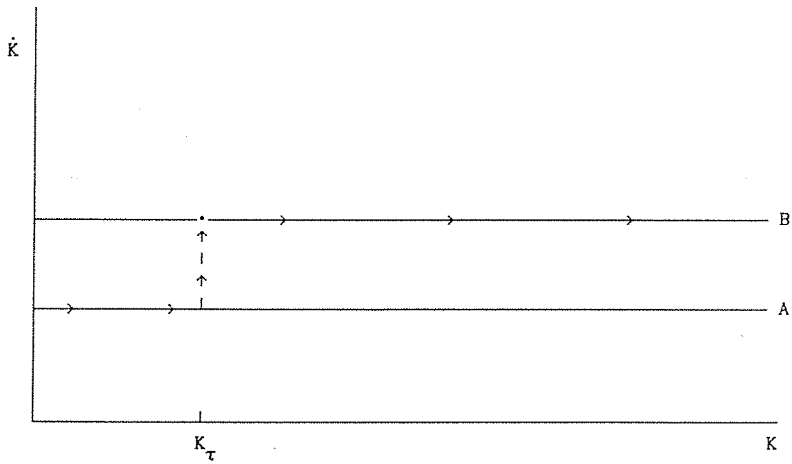


Figure 3.3

($\dot{K}$,K) Phase Diagram: Constant Returns in Production and Net Investment

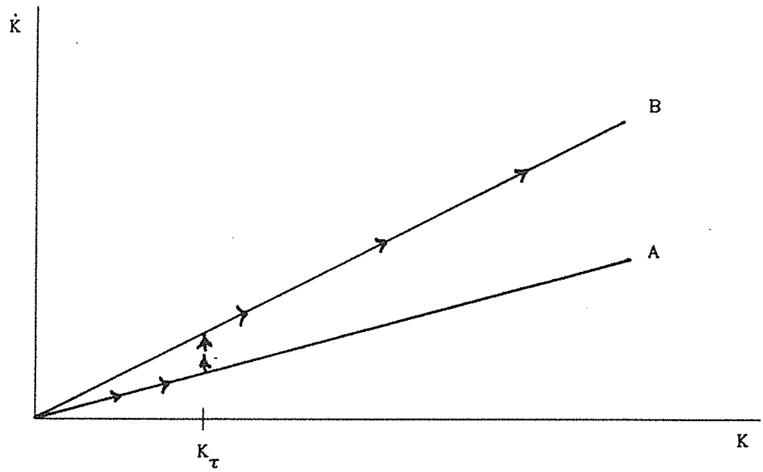
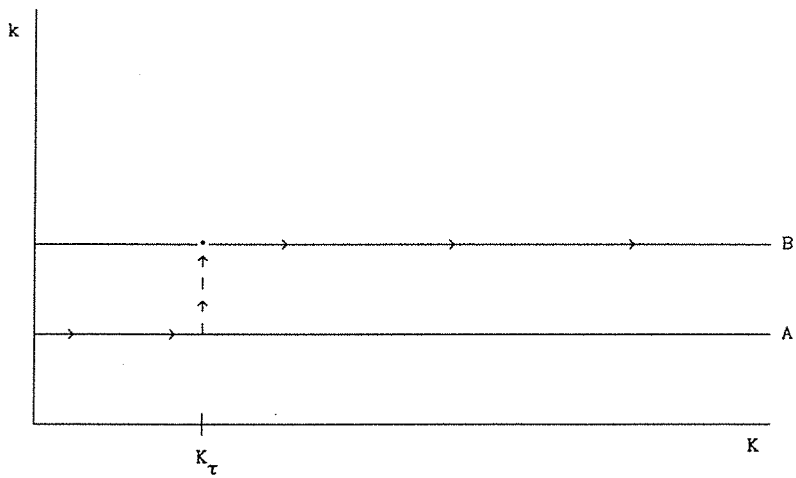


Figure 3.4

(k,K) Phase Diagram: Constant Returns in Production and Net Investment



3.2 A Simple Example with Constant Returns.

A generic example of the model that we have in mind can be posed as the following optimization problem:

Choose paths of gross investment I and capital stock K to maximize

$$\int_{\tau}^{\infty} e^{-r(t-\tau)} \left\{\beta K - \dot{K}.C(\dot{K}/K) - sI\right\} dt \tag{3.2}$$

subject to

$$\dot{K} = I - \delta K$$

$$K(\tau) = K_{\tau} .$$

Define the maximized value as $V(K_{\tau})$. In production of output, characterised by βK, the firm faces constant returns to scale. The marginal product of capital in output, β, could be thought of as a more general function of output prices and variable input prices. Gross investment goods I cost a market price s per unit, but the firm also faces internal costs of adjustment, $C(\dot{K}/K) = C(k)$, say, where internal costs per unit of expansion are a function of the rate of expansion $k = \dot{K}/K$. This function should satisfy $C'(k) > 0$ for $k > 0$, and without loss of generality we can assume $C(0) = 0$. The essential characteristic of problem (3.2) is that the integrand,

$$F(t,K,\dot{K},I) = e^{-r(t-\tau)}\left\{\beta K - \dot{K}C(\dot{K}/K) - sI\right\}$$

is homogeneous of degree 1 (HOD1) in $K,\dot{K},I$, which will ensure that V is linear in K, for constant β and s; or, equivalently, on substituting out $I = \dot{K} + \delta K$ the integrand becomes $e^{-r(t-\tau)}\left\{\beta K - \dot{K}\, C(\dot{K}/K) - s\dot{K} - s\delta K\right\}$, so that $B(K,\dot{K}) = \beta K - \dot{K}C(\dot{K}/K) - s\dot{K} - s\delta K$ is HOD1 in $K,\dot{K}$.

The Hamiltonian for this specification is

$$H = \left\{\beta K - \dot{K}\, C(\dot{K}/K) - s\dot{K} - s\delta K\right\} + \lambda\dot{K}$$

and condition (3.1a) ($H_{\dot{K}} = 0$) implies

$$- C - (\dot{K}/K)\, C' = (s - \lambda)$$

while condition (3.1d) is

$$\dot{\lambda} - r\lambda = - H_K = - \beta - \left(\frac{\dot{K}}{K}\right)^2 C' + s\delta \; .$$

Under static expectations for s, $\beta(P)$, these conditions are satisfied by the existence of a constant rate of growth of capital, k, and a constant shadow price of capital:

$$\dot{\lambda} = 0$$

$$\dot{K} = k.K \text{ , say.}$$

Solving for λ then gives

$$\lambda = \frac{(\beta - s\delta)}{r} + \frac{1}{r}\, k^2 C'(k)$$

and hence k is the solution to

$$r\, C(k) + (r - k)k\, C'(k) = \beta - s(r + \delta) \; . \qquad (3.3)$$

For example, if

$$C(k) = \theta\, k$$

then (3.3) becomes

$$r\, \theta\, k + (r-k)\, k\, \theta = \beta - s(r+\delta)$$

with solution

$$k = r \pm \sqrt{r^2 - [\beta - s(r+\delta)] / \theta}$$

The transversality condition (2.7f) requires $\lim_{t\to\infty} e^{-rt} \lambda K = 0$, i.e. $\lim_{t\to\infty} e^{-rt} e^{kt} = 0$, and hence $k < r$ which suggests choosing the - sign provided the term within the root is positive.

3.3 Generalizing Output and Variable Input Specifications.

In the example of Section 3.2 the term βK could be interpreted as having arisen from a specification of revenue from output Y at price p, pY, and a fixed coefficient production function $Y = aK$, i.e., $\beta = ap$. More generally, for m outputs Y with price vector p, and n variable inputs L with price vector w, technology could be characterized by the set $\mathcal{T}$, where (Y,L,K) is feasible if $(Y,L,K) \in \mathcal{T}$, and constant returns are characterized by $(\eta Y,\eta L,\eta K) \in \mathcal{T}$ iff $(Y,L,K) \in \mathcal{T}$ for $\eta > 0$.

The variable factors L and outputs Y can then be optimized out to define the restricted profit function:

$$E(p,w,K) = \max \left\{ p'Y - w'L : (Y,L,K) \in \mathcal{T} \right\}$$

E(p,w,K) would satisfy the standard conditions of a restricted profit function, (homogeneous of degree 1 in (p,w), increasing in p, decreasing in w, convex in (p,w), increasing in K) and the constant returns to scale assumption on the technology would correspond to the specification[3] $E(p,w,K) = \beta(p,w).K$.

3.4 Generalizing Adjustment Costs.

The term $\dot{K}\, C(\dot{K}/K)$ in the specification (3.2) can be interpreted more generally as the penalty cost of rapidity of expansion, with the cost per unit of $\dot{K}$ an increasing function of the rate of expansion, $\dot{K}/K$. Thus this term is merely the cost function associated with "producing" the level of expansion $\dot{K}$. One possible generalization would be to model the "output" of expansion, $\dot{K}$, as being produced by the inputs K, L and I according to some production function. Then the minimum cost of "producing" $\dot{K}$ conditional on levels of K and prices of variable inputs would be characterized by the (restricted) cost function

$$C(\dot{K},w,s,K) \ .$$

There seems to be no consensus in the literature as to whether this cost function should be specified to be an explicit function of net investment, $\dot{K}$, a case referred to as internal costs of adjustment, or of gross investment, I, which is typically referred to as external costs of adjustment, and modelled by allowing the price of new investment goods, s, to be an increasing function of the level of gross investment, s(I). Within the context of a CGE model, it would seem more appropriate to model the supply of investment goods separately, and hence in what follows we will use as an example the case of internal adjustment costs. Then C can be structured as

$$C(\dot{K},w,s,K) \ = \ \bar{C}(\dot{K},w,K) + s.(\dot{K}+\delta K) \ .$$

C and $\bar{C}$ would have standard properties in w (increasing, convex and HOD1 in w) and variable factors used in installing $\dot{K}$ would be given by Shephard's Lemma

$$L_i(\dot{K},w,s,K) \ = \ \bar{C}_{w_i}$$

Total demand for variable factors would be

$$L_i(\dot{K},w,s,Y,K) = -E_{w_i} + \bar{C}_{w_i} .$$

It could, however, be argued that $\bar{C}$ should be a function of the levels of the outputs Y, or, for simsilar reasons, that E should be a function of the output of $\dot{K}$.

With the generalizations introduced above, it is then a short step to model the technology as one of using quasi-fixed inputs K and variable inputs L to produce two types of output, the output vector Y sold at market prices p, and the output "expansion of the capacity of the quasi-fixed factor", $\dot{K}$. Treadway (1971) has argued against the implied separability that would arise by treating the two types of output separately.

Thus in general the technology could be characterized by the set $\mathcal{T}$ where

$$(Y,L,K,\dot{K}) \in \mathcal{T}$$

describes the outputs Y and $\dot{K}$ technically possible with the inputs K,L, and the firm's problem is to choose paths for Y, L, K, $\dot{K}$ and I to

$$\text{maximize} \int_{\tau}^{\infty} e^{-r(t-\tau)} \left\{p'Y - w'L - sI\right\} dt$$

$$\text{s.t. } (Y(t),L(t),K(t),\dot{K}(t)) \in \mathcal{T}$$

$$\dot{K} = I - \delta K$$

$$K(\tau) = K_{\tau} .$$

At each point in time t, the perfectly variable outputs Y and inputs L can be optimized out to give the (twice) restricted "profit" function

$$G(p,w,K,\dot{K}) = \max_{Y,L} \left\{ p'Y - w'L : (Y,L,K,\dot{K}) \in \mathcal{T} \right\} .$$

G will inherit the "standard" profit function properties in prices: G should be HOD1 in p and w, non-decreasing in p, non-increasing in w, convex in (w,p), with the $(K,\dot{K})$ conditioned supplies and demands given by Hotelling's Lemma:

$$Y_i = G_{p_i} \tag{3.4a}$$

$$L_j = -G_{w_j} \tag{3.4b}$$

Equations (3.4a) and (3.4b) give the optimal supplies of output (Y_i) and demands for variable inputs (L_j) as functions of prices (p,w), the level of capital stock (K) and the level of net investment $(\dot{K})$. The costs of adjustment specification would be captured by requiring G decreasing and convex in $\dot{K}$, and G would be increasing in K. The corresponding Hamiltonian would be

$$H = G(p,w,K,\dot{K}) - s(\dot{K} + \delta K) + \lambda\dot{K}$$

with first-order conditions:

$$G_{\dot{K}} - s + \lambda = 0 \tag{3.4c}$$

$$\dot{\lambda} - r\lambda = s\delta - G_K . \tag{3.4d}$$

Corresponding to the three examples in Section (3.1), three special cases of this model may be characterised. Firstly, if the technology is such that G is strictly concave in $(K,\dot{K})$, for static expectations of p,s,w,r (3.4c,d) will be satisfied by $\dot{K} = 0$, $K = K^*$ and locally net investment will be of the flexible accelerator form

$$\dot{K} = M(K - K^*) .$$

Second, if G is separable in $K,\dot{K}$ of the form

$$G = \beta(p,w).K - C(\dot{K},p;w)$$

with $C(\dot{K})$ strictly convex in $\dot{K}$, for static expectations (3.4c,d) will be satisfied by $\dot{K}$ = constant. Third, if the technology is subject to constant returns to scale $((\eta Y,\eta L,\eta K,\eta\dot{K}) \in \mathcal{T}$ if $(Y,L,K,\dot{K}) \in \mathcal{T}$ for $\eta > 0)$, then G is HOD1 in $K,\dot{K}$ and

$$G(p,w,K,\dot{K}) = G(p,w,1,\dot{K}/K).K$$

$$= g(p,w,k).K ,$$

with g decreasing and concave in k. The $(K,\dot{K})$ conditioned supplies and demands are

$$y_i = Y_i / K = g_{p_i} \tag{3.5a}$$

$$\ell_j = L_j / K = -g_{w_j} , \tag{3.5b}$$

the Hamiltonian is

$$H = \left\{g(p,w,k) - sk - s\delta + \lambda k\right\} K$$

$$= h(p,w,k,\lambda) . K$$

where the control variable is now k, i.e., $\dot{K} = kK$. To maximize H (or h) the necessary condition is

$$g_k - s + \lambda = 0 \tag{3.5c}$$

and the transition equation for λ is

$$\dot{\lambda} - r\lambda = - h$$

$$= -\left\{g - sk - s\delta + \lambda k\right\} . \tag{3.5d}$$

With static expectations for p,s,w,r, these conditions are satisfied by

$$\dot{\lambda} = 0$$

i.e., with λ, k solutions to

$$(r-k)\, g_k + g = s(r+\delta) \tag{3.6a}$$

$$\lambda = (g - sk - s\delta) \,/\, (r-k) \tag{3.6b}$$

providing solutions for λ^{ss} and k^{ss}, the steady-state solutions for λ and k.

3.5 <u>A Once Restricted Profit Function</u>.

In terms of the primal specification of technology, $\mathcal{T}$, the appropriate specification of the Hamiltonian is

$$H = p'Y - w'L - s\dot{K} - s\delta K + \lambda\dot{K}$$

and treating Y, L and $\dot{K}$ as controls requires that H is to be maximized w.r.t. Y, L and $\dot{K}$, and subject to the technology $\mathcal{T}$. Thus if the "price" of $\dot{K}$ is defined as λ-s, $\dot{K}$ can be treated as an output in a manner symmetric to Y. Hence define the once restricted profit function:

$$\hat{H}(p,w,\lambda-s,K) = \max H \text{ s.t. } (Y,L,\dot{K},K) \in \mathcal{T} .$$

$\hat{H}$ is hence analogous to a standard restricted profit function, and will be HOD1 in p,w,λ-s, non-decreasing in p,λ-s, non-increasing in w and convex in p,w,λ-s. With the technology specified by the functional form of $\hat{H}$, the (λ,K) conditioned output supply, variable input demand, and net investment demand functions are given by Hotelling's Lemma:

$$Y_i = \hat{H}_{p_i} \tag{3.7a}$$

$$L_j = -\hat{H}_{w_j} \tag{3.7b}$$

$$\dot{K} = \hat{H}_{\lambda-s} \tag{3.7c}$$

with transition equation for λ

$$\dot{\lambda} - r\lambda = -\hat{H}_K . \tag{3.7d}$$

Under the restriction of constant returns to scale technology, H can be written as

$$H = (p'y - w'\ell - sk - s\delta + \lambda k)K$$

and hence $\hat{H}(p,w,\lambda-s,K) = \hat{h}(p,w,\lambda-s).K$. System (3.7) then becomes

$$y_i = \hat{h}_{p_i} \tag{3.8a}$$

$$\ell_j = -\hat{h}_{w_j} \tag{3.8b}$$

$$k = \hat{h}_{\lambda-s} \tag{3.8c}$$

and $$\dot{\lambda} - r\lambda = -\hat{h} . \tag{3.8d}$$

This sequence of specifications brings the problems with the solution of such a model into context. Three related issues are: the way in which technology is specified, the structure on technology (such as constant returns to scale); and expectations of P, r and T. In principle, for any specification of technology (say as a functional form for B( )), and for any specification of expectations, the system

(3.1a-f) could be solved. Of this system, (3.1a-c) play a special role, in that only current values of variables appear - because the solutions are conditional on λ. Thus solutions of (3.1a-c) are independent of expectations of future variables, and if they can be solved explicitly then the technology can be explicitly represented in terms of a once restricted profit function. More generally, however, there will exist specifications of technology in terms of the once restricted profit function for which explicit D( ) or G( ) functions cannot be derived.

Solution of (3.1a) and (3.1b) then allows the intertemporal nature of the problem to be captured in a pair of differential equations and two endpoint conditions. Again, numerical solution is possible, but analytical solution requires further structure on the problem, such as, for example, static expectations and constant returns to scale. But in terms of the specification of technology, representation in terms of a once restricted profit function would appear to be the most general approach, and a possible approach to the estimation of parameters for such a specification in the static expectations case is considered in Appendix 1.

4. Econometric Estimation of Parameters

4.1 Introduction

In order to focus on the issues that arise in the estimation of the parameter vector θ, consider the problem faced by the firm in a form which is a generalization of the model specified in terms of a twice restricted profit function. The firm's problem is to choose the time path of net investment to maximize

$$\int_{\tau}^{\infty} e^{-R(t)}\left\{G\left[p(t),w(t),T(t),K(t),\dot{K}(t);\theta\right] - s(t)\left[\dot{K}(t) + \delta K(t)\right]\right\}dt \;. \tag{4.1}$$

subject to $K(\tau) = K_\tau$ given,

where $R(t) = \int_{\tau}^{t} r(u)du$, with $r(u)$ the required rate of return at time u. If $\{P_\tau\}$ represents the paths of $p(t)$, $w(t)$, $s(t)$ and $r(t)$ for $t \geq \tau$, and $\{T_\tau\}$ represents the path of $T(t)$ for $t \geq \tau$, the maximized value of (4.1) can be represented by the value functional $V(K_\tau,\{P_\tau\},\{T_\tau\})$. In general, $\{P_\tau\}$ and $\{T_\tau\}$ represent expectations held at time τ, based on the information set available at τ, say Ω_τ. For simplicity, the problem has been posed in terms of net investment, $\dot{K}(t)$. The identities $\dot{K}(t) = I(t) - \delta K(t)$ or $I(t) = \dot{K}(t) + \delta K(t)$ allow simple transition between net investment or gross investment specifications.

The use of a variable discount factor has only a minor impact on the necessary conditions (but a substantial impact on their solution). To derive the necessary conditions, form the present value Hamiltonian

$$\tilde{H} = e^{-R(t)}\left\{G(p,w,K,\dot{K}) - s(\dot{K}+\delta K)\right\} + \Lambda(t)\dot{K}$$

with first order conditions:

$$\tilde{H}_{\dot{K}} = 0, \quad \text{i.e.} \quad e^{-R(t)}(G_{\dot{K}} - s) + \Lambda = 0 \tag{4.2a}$$

$$\dot{\Lambda} = -\tilde{H}_K \quad \text{i.e.} \quad \dot{\Lambda} = -e^{-R(t)}\{G_K - s\delta\} \;. \tag{4.2b}$$

In terms of the current value Hamiltonian $H = e^{R(t)}\tilde{H}$ and the current value co-state $\lambda = e^{R(t)}\Lambda$, $\dot{\lambda} = e^{R(t)}\, r(t)\Lambda + e^{R(t)}\dot{\Lambda}$ and hence (4.2a,b) are replaced by

$$(G_{\dot{K}} - s) + \lambda = 0 \tag{4.3a}$$

$$\dot{\lambda} = r\lambda - (G_K - s\delta) \tag{4.3b}$$

with corresponding initial condition

$$K(\tau) = K_\tau \tag{4.3c}$$

and transversality condition

$$\lim_{t\to\infty} e^{-R(t)} \lambda(t) K(t) = 0 . \tag{4.3d}$$

In the literature, at least four approaches to the estimation of (θ,δ) can be identified. These approaches may be distinguished on a number of criteria, such as: data requirements; allowance for expectations; generality of functional form; and degree of exploitation of restrictions implied by optimization. For exposition, these approaches will be referred to as: "second generation dynamics"; "closed form solutions", "q-theory"; and "Euler equation estimation". Each of these approaches will be considered in turn, with an attempt to evaluate them in terms of the above criteria.

4.2 Second Generation Dynamics

For arbitrary expectations of $\{P_\tau\}$, $\{T_\tau\}$, then conditional on the level of $K(t)$ and $\dot{K}(t)$, the optimal $(K,\dot{K})$ conditioned levels of the variable inputs and outputs are given by Hotelling's Lemma:

$$Y_i(p,w,T,K,\dot{K}) = G_{p_i}(p,w,T,K,\dot{K}) \tag{4.4a}$$

$$L_j(p,w,T,K,\dot{K}) = - G_{w_j}(p,w,T,K,\dot{K}) \tag{4.4b}$$

"Profits", defined as dividends plus retained earnings, (Π), satisfy

$$\Pi = G(p,w,T,K,\dot{K}) - s(\dot{K} + \delta K) \tag{4.4c}$$

In system (4.4) all variables are dated at time t. As a system, (4.4) would allow estimation of the parameters of G( ), designated θ, and of δ. A recent example of such a procedure is Galeotti (1990), who used a quadratic functional form for G and the implied system (4.4) to estimate adjustment cost parameters. Galeotti attributes the terminology "second generation dynamics" to Berndt, Morrison and Watkins (1981).

In terms of the criteria introduced in Section 4.1, this approach would seem to be: not very demanding in its use of data; not restrictive in the modelling of expectations; applicable to any specification of technology in terms of a twice restricted profit function; but utilizing none of the intertemporal optimizing conditions. In addition, the potential simultaneity of (at least) $\dot{K}$ is a problem in estimation, and the need to appeal to instrumental variables estimation will further reduce the efficiency of estimation.

In general, it is difficult to see how an econometric specification which conditions on the level of investment can be very informative of the parameters of the investment process.

4.3 Closed Form Solutions

In principle, at least, for any functional form specification of G (or equivalently of the technology $\mathcal{T}$) system (4.3) generates a closed loop or feed-back solution, of the form

$$\dot{K}(t) = \dot{K}^{c}(K(t); \{P_\tau\},\{T_\tau\}; \theta) \tag{4.5a}$$

(and corresponding solution for $\lambda(t)$,

$$\lambda(t) = \lambda^{c}(K(t); \{P_\tau\},\{T_\tau\}; \theta)\ ) \tag{4.5b}$$

giving, at time $t = \tau$, the decision rule at time t conditional on information available at time t.

Equation (4.5a), together with (4.4a,b) would exploit all of the information available from the optimization paradigm. The corresponding system could be estimated simultaneously, or alternatively (4.5a) could be used to eliminate $\dot{K}$ from (4.4a,b) to give a complete (reduced form) system of equations analogous to those typically estimated for static models. (Again, analogously with static systems, (4.4c) could also be included, but is essentially redundant, and would merely introduce a singularity into the system). The above system is also in a form which could easily be integrated into an intertemporal computable general equilibrium model.

The problem with this approach is that the closed form solution (4.5a) is available only under very restricted specifications on the form of G and/or the specification of expectations. There are two subclasses of approach, one based on more general expectations but restricted functional form, and the other based on more general functional form, but restricted (in general, static) expectations.

(i) Quadratic Functional Forms, Rational Expectations.

If the expectation formation mechanism is specified in terms of restricted vector difference equations, and if the profit function is quadratic, the linear-quadratic nature of the optimization problem allows, via certainty equivalence, the derivation of closed form decision rules. These decision rules can then be estimated jointly with the system of constant coefficient linear difference equations with restrictions across equations, to give the parameter estimates consistent with rational expectations. For examples of this approach, see Kennan (1979), Hansen and Sargent (1980, 1982), Nerlove, Grether and Carvalho (1979), Sargent (1979). A simple example of this approach is given in Section 4.6.

In terms of the criteria of Section 4.1, this approach uses available data, allows for rational expectations (although with a restricted generating mechanism), is quite restricted in functional form, but does exploit all restrictions implied by the optimization.

(ii) Static Expectations Close Loop Solutions.

Closed form solutions have typically only been derived under the assumption of static expectations for p, w, s and r, and particular functional forms for G( ). These functional forms are typified by the three examples of section 3.1. The most common empirical example is probably case (a), the flexible accelerator, where the form $\dot{K} = \mu(K-K^*)$ is derived as the local behaviour (about $\dot{K} = 0$, $K = K^*$) based on a quadratic approximation to G. This gives the local behaviour of the dependence of μ and $\dot{K}$ on p, w, s and r.

Case (b) can be generalized as in Section 3.2 to any specification of the form $G(p,w,s,\dot{K},K) = \beta(p,w)K - C(p,w,s,\dot{K})$ for which the optimal solution for $\dot{K}$ satisfies

$$C_{\dot{K}}(p,w,s,\dot{K}) = [\beta(p,w) - s\delta] / r . \qquad (4.6)$$

Equation (4.6) is analytically tractable for C( ) specified as

$$C = a\dot{K}^{\phi} + b\dot{K}$$

where a and b are functions of p, w and s, but an equation like (4.6) could be used to generate parameter estimates for any specification of C convex in $\dot{K}$. (See Appendix 1 for a general discussion of the estimation of equations defined implicitly as in (4.6).) However, as noted earlier, such solutions for $\dot{K}$ are by construction independent of K, and hence are probably of limited interest.

Case (c) of Section 3.1 can be similarly generalized for the case in which G is HOD1 in $(K,\dot{K})$:

$$\begin{aligned} G(p,w,T,K,\dot{K}) &= G(p,w,T,1,\dot{K}/K).K \\ &= g(p,w,T,k).K \ , \end{aligned}$$

and from Section 3.4, k is a solution to

$$(r-k)\ g_k(p,w,T,K) + g(p,w,T,k) = s(r+\delta). \tag{4.7}$$

It is of interest to derive (4.7) in an alternative way. Since G is HOD1 in K, $\dot{K}$, if $\dot{K}(t)$ is an optimal path when K(t) satisfies $K(\tau) = K_\tau$, then $A\dot{K}(t)$ must be optimal for an initial condition of AK_τ, where A > 0 is arbitrary. Hence at each point in time k(t) is to be chosen, and under static expectations the planned path of k(t) is constant. Hence

$$V(K_\tau,p,w,s,r,T) =$$

$$\underset{k}{\text{Max}} \int_\tau^\infty e^{-r(t-\tau)}\left\{g(p,w,T,k) - s(k+\delta)\right\} K_\tau\ e^{k(t-\tau)}\ dt$$

$$= K_\tau\ .\ \underset{k}{\max} \frac{g(p,w,T,K) - s(k+\delta)}{r-k}$$

$$= v(p,w,s,r,T)K_\tau \ ,$$

where $v(p,w,s,r,T) = V(K_\tau,p,w,s,r,T)\ /\ K_\tau$.

Hence the optimal investment rate k is a solution to the maximization problem

$$v(p,w,s,r,T) \;=\; \max_{k} \frac{g(p,w,T,k) - s(k+\delta)}{r-k} \;. \qquad (4.8)$$

In principle, there is a duality between functional forms for g( ) and functional forms for v( ), with the dual optimization problem specified by

$$g(p,w,T,k) \;=\; \min_{s,r,\delta} \left\{(r-k)v(p,w,s,r,T) + s(k+\delta)\right\} \qquad (4.9)$$

From the envelope theorem, the analogue of Hotelling's Lemma is

$$k \;=\; \frac{r\,v_s + \delta}{v_s - 1} \qquad (4.10)$$

This result is the specialization to the case of constant returns to scale of a duality between V and G, and the corresponding analogue of Hotelling's Lemma, proposed in Cooper and McLaren (1980, 1987a), and developed and applied in Epstein (1981) and Epstein and Denny (1983). Unfortunately, the duality between V and G is not complete to the extent of allowing specific alternative functional forms for V, and the use of flexible approximate functional forms for V by the above authors detracts from its general appeal. The special case represented by (4.8) - (4.10) may provide an avenue for more useful functional forms.

4.4 q-theory of Investment

Equation (4.3a) is the first order condition corresponding to the maximization of the Hamiltonian H: the optimal level of $\dot{K}$ maximizes

$$H \;=\; G(p,w,T,K,\dot{K}) - s(\dot{K} + \delta K) + \lambda\,\dot{K} \;. \qquad (4.11)$$

For arbitrary expectations on the future values of p, w, s, T and r, the costate variable λ represents the shadow price of an extra unit of capital, and the first order condition for the maximization of H requires that this shadow price be equated with the marginal cost of an extra unit of capital, $s+(-G_{\dot{K}})$ (recall that $G_{\dot{K}}$ is negative). Hence, if λ were observable, then even in the case of arbitrary expectations there would be no need to solve an intertemporal problem - instead, all that is required is the solution of (4.11).

It is a well-known result in control theory that $\lambda = \lambda(K_\tau, \{P_\tau\}, \{T_\tau\}) = V_K(K_\tau, \{P_\tau\}, \{T_\tau\})$ and hence in general, λ will not be observable. In the special case in which V is linear in K, however,

$$\lambda = \frac{V}{K} = v$$

and λ can in principle be measured in terms of the average stock market valuation of capital. Following Tobin (1969) and Hayashi (1982), it is common to work in terms of the ratio of the average value of capital to replacement cost,

$$q = \lambda/s = \frac{V}{sK} \tag{4.12}$$

and to define the optimal level of net investment as the solution to

$$-\frac{1}{s} G_{\dot{K}}(p,w,T,K,\dot{K}) = q - 1 \tag{4.13}$$

A sufficient condition for V to be linear in K is that G be HOD1 in $K,\dot{K}$. Thus typical applications of q theory choose a functional form for G that is HOD1 in $K,\dot{K}$ and for which (4.13) has an explicit solution for $\dot{K}$, say

$$\dot{K} = \dot{K}^q(p,w,T,K,s,q) \tag{4.14}$$

A leading example of such an application is Summers (1981). The methodology of Appendix 1 suggests that the necessity for an explicit solution of (4.13) is unduly restrictive in the allowable specifications of G( ). Further, the development of the once restricted profit function in Section 3.4 suggests that this would be an appropriate way to specify technology if the q approach were to be considered.

The major problem with q-theory is its demands on the data. There is a serious question as to whether stock market data may be too distorted by noise to give an acceptable representation of the underlying value of q. In application, there must also be serious questions as to the potential endogeneity of q, being jointly determined in the optimization process with $\dot{K}$, and hence the typically simplistic estimation of (4.13) may suffer simultaneity bias.

4.5 The Euler Equation Approach

The emphasis in this approach is on the specification of expectations on $\{P_\tau\}$, $\{T_\tau\}$ according to the rational expectations of these variables conditional on the information set at time τ, Ω_τ. This idea is most easily expressed in a discrete time framework, for which the analogue to (4.1) is that the firm chooses the path of K_t, $t = \tau+1, \tau+2, \ldots$ to maximize

$$\underset{\tau}{E} \sum_{t=\tau}^{\infty} R_t \left\{ G\left(p_t, w_t, K_t, K_{t+1} - K_t\right) - s_t \left(K_{t+1} - K_t + \delta K_t\right)\right\} \qquad (4.15)$$

where $R_t = \prod_{i=1}^{t-\tau} \left(1+r_{\tau+i}\right)^{-1}$,

and $\underset{\tau}{E}$ denotes the expectation conditional of Ω_τ. At time τ, all variables dated τ or earlier are known, and the choice of optimal investment at time τ is equivalent to the choice of $K_{\tau+1}$, for which the first-order necessary condition ("Euler equation") is

$$E_\tau\left[R_\tau\left\{G_{\dot{K}}(\tau) - s_\tau\right\} + R_{\tau+1}\left\{G_K(\tau+1) - G_{\dot{K}}(\tau+1) + s_{\tau+1}(1-\delta)\right\}\right] = 0 \quad (4.16)$$

where $G(t)$ is short-hand for $G(p_t, w_t, K_t, K_{t+1} - K_t)$ and $\dot{K}$ refers to $K_{t+1} - K_t$.

Because (4.1) holds in expectation conditional on Ω_τ, replacement of unknown future variables by their actual values gives

$$R_\tau\left\{G_{\dot{K}}(\tau) - s_t\right\} + R_{\tau+1}\left\{G_K(\tau+1) - G_{\dot{K}}(\tau+1) + s_{\tau+1}(1-\delta)\right\} = u_{\tau+1} \quad (4.17)$$

where $E(u_{\tau+1}|\Omega_\tau) = 0$,
an equation which may be estimated using Hansen and Singleton's (1982) Generalized Method of Moments estimator (GMM).

In order to apply this estimator to variables which are more likely to satisfy the required stationarity properties, it is probably more acceptable to write (4.16) as

$$E_\tau\left[1 + \frac{R_{\tau+1}}{R_\tau} \cdot \frac{G_K(\tau+1) - G_{\dot{K}}(\tau+1) + s_{\tau+1}(1-\delta)}{G_{\dot{K}}(\tau) - s_\tau}\right] = 0 \quad (4.18)$$

This approach will allow the consistent estimation of the parameters θ for a model in which the functional form of $G(\ )$ is quite general, and for a model in which the expectations formation mechanism for future values of variables is quite general. Its drawback is that (4.16) is a necessary condition for the optimality of the path of K_t, but this necessary condition is potentially consistent with a number of other (non-optimal) paths. In particular, no transversality condition has been imposed to eliminate those non-optimal paths. These points are elaborated upon in the example of the next section.

4.6 An Evaluation of the Alternative Estimators

It is convenient to compare those estimators for a simple model for which solutions are known, the linear-quadratic adjustment cost model. For the purposes of this section, we will use the notation of Gregory, Pagan and Smith (1990). The decision-maker attempts to choose a time path for y_s to track a target y^*_s to minimize

$$E_t \sum_{s=t}^{\infty} \beta^{s-t}\left[\delta\left(y_s - y^*_s\right)^2 + \left(y_s - y_{s-1}\right)^2\right] \tag{4.19}$$

where the target is $y^*_s = x'_s\theta + e_s$, e_s is a white-noise error known to the decision-maker (but not the econometrician) at time s, and the expectation is conditional on $\mathcal{F}_t = \left\{e_t,\ y_{t-j},\ x_{t-j+1}\right\}_{j=1}^{\infty}$.

The Euler equation is

$$\beta\ E_t\ y_{t+1} - (1+\beta+\delta)y_t + y_{t-1} + \delta y^*_t = 0\ . \tag{4.20}$$

If λ is the stable root of the characteristic equation $\beta z^2 - (1+\beta+\delta)z + 1 = 0$, the forward solution to the Euler equation is

$$y_t = \lambda\ y_{t-1} + (1-\lambda)(1-\beta\lambda)\ E_t \sum_{s=t}^{\infty} (\beta\lambda)^{s-t}\ y^*_s \tag{4.21}$$

which can be used directly only if the expectations for $E_t y^*_s$ can be specified. In general, however, it is to be expected that the closed loop solution would be of the form

$$y_t = f\left(x_t, x_{t-1}, \ldots, y_{t-1}, y_{t-2}, \ldots, e_t\right) . \tag{4.22}$$

Estimation of the Euler equation involves replacing $E_t \; y_{t+1}$ by $y_{t+1} + \eta_{t+1}$ where $E_t \; \eta_{t+1} = 0$, and y_t^* by $x_t'\theta + e_t$ to give

$$\beta y_{t+1} - (1+\beta+\delta)y_t + y_{t-1} + \delta x_t'\theta = u_{t+1} \tag{4.23}$$

$$\text{where } u_{t+1} = -\left(\beta\eta_{t+1} + \delta e_t\right).$$

If an assumption is made as to how x_t is generated, then the expectation y_s^* can be substituted out of (4.21). For example, if

$$(1-\rho L)x_t = \varepsilon_t, \quad E_{t-1}\varepsilon_t = 0 \tag{4.24}$$

then the solution can be written in ECM form as

$$\begin{aligned} \Delta y_t = \; & (\lambda-1)(y_{t-1} - x_{t-1}\theta) \\ & + (1-\lambda)\,\theta\left[(1-\rho\beta\lambda)^{-1}(1-\beta\lambda)x_t - x_{t-1}\right] \\ & + (1-\beta\lambda)(1-\lambda)\,e_t . \end{aligned} \tag{4.25}$$

Equation (4.25) is the closed loop solution, and efficiency would suggest that it be estimated jointly with equation (4.24). (This is the methodology typically used in the approach of Section (4.3(i)). However, the important point is the comparison of (4.25) with (4.23). In general, estimation of (4.23) will be less efficient than estimation of (4.25) for (at least) two reasons:

(a) equation (4.25) is parameterized on (λ,β,θ), while (4.23) is parameterized on (δ,β,θ). Since λ is a constrained root of the characteristic equation, this constraint $(\lambda<1)$ can be imposed on (4.25) to gain efficiency. This reflects the fact that while (4.20) is a necessary condition for optimality, it is only one necessary condition. The constraint $\lambda<1$ corresponds to a transversality condition being imposed. (This point is made in the literature, see for example Shapiro (1986).)

(b) After adjusting for timing, (4.23) explains y_t in terms of y_{t-1}, y_{t-2}, x_{t-1}; in constrast (4.25) explains y_t in terms of y_{t-1}, x_t, and x_{t-1}. Thus (4.23) does not use the most recent information available on x_t, and instead apparently "proxies" x_t with y_{t-2}! While this is consistent with the usual justification for an Euler equation - that along an optimal path it must be the case that conditional on the knowledge contained in y_{t-1}, the planned change to y_t is random (c.f. Hall (1978)), - the point is that new information does become available that may be incorporated in y_t at time t. An Euler equation characterises planned movement through time, i.e. the relation of $E_t y_{t+1}$ to y_t, but does not capture the actual movement through time, the extent to which y_{t+1} changes from y_t due to the information contained in x_t (c.f. (4.25)). It is this explanation that is probably responsible for the poor simulation behaviour of the Euler equation type estimators in Gregory, Pagan and Smith (1990).

5. Conclusion

The paradigm proposed by Wilcoxen (1989) for the incorporation of adjustment cost investment models into an intertemporal computable general equilibrium model is potentially rich in applications. Section 3 of this paper has used duality theory in order to extend the specifications of technology to which this approach may be applied, and Section 4 has considered some of the possibilities that exist for the econometric estimation of the parameters of such a model. It now remains to gain experience with the application of some of these models and their estimated parameters to evaluate the alternative approaches.

Appendix 1

Econometric Estimation of Implicit Solutions to Optimization Models

In general, the first order conditions of the optimization models considered in this paper lead to a system of nonlinear simultaneous equations of the form

$$f_{it}(y_t, x_t, \theta) = 0 \qquad \begin{array}{l} i = 1,2,\ldots,n \\ t = 1,2,\ldots,T \end{array} \tag{A.1}$$

where y_t is an n-vector of decision variables at time t, x_t is an n-vector of predetermined variables at time t, and θ is a vector of parameters.

If the system of equations (A.1) allows an explicit analytical solution, say

$$y_{it} - g_{it}(x_t, \theta) = 0 \tag{A.2}$$

the standard approach of FIML estimation is to replace (A.2) by

$$y_{it} - g_{it}(x_t, \theta) = u_{it} , \tag{A.3}$$

assume

$$u_t \sim N(0, \Sigma) ,$$

and define the FIML estimator of θ as those values of θ that maximize the log likelihood

$$L_1^* = -\frac{T}{2} \log |\Sigma| - \frac{1}{2} \sum_{t=1}^{T} \left[y_t - g_t(x_t,\theta)\right]' \Sigma^{-1} \left[y_t - g_t(x_t,\theta)\right] . \tag{A.4}$$

or, equivalently, the concentrated log likelihood

$$L_1^* \;=\; -\frac{T}{2}\log\left|\frac{1}{T}\sum_{t=1}^{T}\left[y_t - g_t(x_t,\theta)\right]\left[y_t - g_t(x_t,\theta)\right]'\right| \; . \tag{A.5}$$

In (A.3), the variables y_{it} may have been appropriately transformed to justify parameterizing on Σ - for example, input demands are often normalized as shares.

If no such analytical solution is available, the standard procedure would seem to be to replace (A.1) by

$$f_{it}(y_t,x_t,\theta) \;=\; v_t \; , \tag{A.6}$$

assume

$$v_t \sim N(0,\Omega)$$

and define the FIML estimator of θ as those values of θ that maximize the log-likelihood

$$L_2^* \;=\; -\frac{T}{2}\log\mid\Omega\mid + \sum_{t=1}^{T}\log\|\,\partial f_t \,/\, \partial y_t'\,\| - \frac{1}{2}\sum_{t=1}^{T} f_t'\,\Sigma^{-1}\,f_t \; , \tag{A.7}$$

or, equivalently, the concentrated log-likelihood

$$L_2 \;=\; \sum_{t=1}^{T}\log\|\,\partial f_t \,/\, \partial y_t'\,\| - \frac{T}{2}\log\left|\frac{1}{T}\sum_{t=1}^{T} f_t f_t'\right| \tag{A.8}$$

In the linear case in which (A.1) may be represented as

$$Ay_t + Bx_t \;=\; u_t$$

the models are related by $u_t = Av_t$, $\Sigma = A\Omega A'$ and the FIML estimators of θ defined by (A.4) and (A.7) coincide. In the case of a nonlinear function f_t, however, (A.3) and (A.6) are mutually inconsistent (parameterizing on Σ is not consistent with parameterizing on Ω), and in general parameter estimates generated by maximizing (A.4) and (A.7) will not coincide.

In fact, for the types of models considered in this paper, estimators based on (A.6) are not well defined. For optimization models, the set of equations (A.1) correspond to n first order necessary conditions for the n decision variables y_{it}, and such first-order conditions can be written in a number of ways that convey equivalent information - in fact, (A.1) and (A.2) can be thought of as two extremes, with various intermediate possibilities. It could, in fact, be argued that (A.2) is the only objective representation of the first-order conditions.

The fact that an analytical representation for (A.2) is not available is really only of minor consideration. For non-linear models, an iterative procedure is required to maximize (A.4), and the only role of (A.2) is to compute the value of y_{it}, say $\hat{y}^r_{it}$, implied by x_t and $\hat{\theta}^r$ at the r^{th} iteration, directly evaluated as

$$\hat{y}^r_{it} = g_{it}\left(x_t, \hat{\theta}^r\right) .$$

But within each iteration, it is a minor numerical problem to compute $\hat{y}^r$ from (A.1) as the solution to

$$f_{it}\left(\hat{y}^r, x_t, \hat{\theta}^r\right) = 0 .$$

At worst, this procedure may require the numerical solution of n non-linear equations at each iteration, but, depending upon the specification of the problem, the dimensionality of this numerical solution may be substantially less than n. For example, in the context of consumer demand models, for an implicit representation of an indirect utility function (as in, for example, Hanoch (1975)) the dimension of this numerical problem is 1. An explicit representation of (A.2) is a case of zero dimensionality.

Within the context of the current paper consider the constant returns to scale once restricted profit function

$$\hat{h}(p, w, \lambda - s, \theta)$$

with first-order conditions

$$y_i = \hat{h}_{p_i} \tag{A.7a}$$

$$\ell_j = \hat{h}_{w_j} \tag{A.7b}$$

$$k = \hat{h}_{\lambda - s} \tag{A.7c}$$

$$\dot{\lambda} = r\lambda - \hat{h} \tag{A.7d}$$

Under static expectations, (d) is replaced by

$$0 = r\lambda - \hat{h}(p, w, \lambda - s; \theta) \tag{A.7d'}$$

and for each $\hat{\theta}^r$, at each t, (A.7d') can be solved for $\lambda_t(p, w, s, \hat{\theta}^r)$ to allow closed form solutions for (A.7 a-c). (The fact that λ_t varies over t merely reflects the fact that static expectations is inconsistent with sample variation). Hence the model of Section 3.5, while designed for the convenience of solution of dynamic CGE models, provides only minor problems at the estimation stage (provided, of course, that at the estimation stage one is willing to accept the myopic assumption of static expectations required in all non-quadratic closed form solutions).

Footnotes

1. A reference for the methods of calculus of variations and optimal control is Kamien and Schwartz (1981).

2. In a static profit maximizing model, the case of diminishing returns to scale allows determination of a desired level of capital, and hence the level of investment as the difference between desired and actual capital. In a dynamic model, this approach to investment can be maintained in discrete time (see Brechling (1975)), but for a continuous time specification the rate of investment is unbounded. This property is usually considered *a priori* unacceptable in such models. By contrast, with convex adjustment costs the rate of investment is well defined, i.e. determinate, even in the case of constant returns to scale.

3. A standard reference on the properties of cost and profit functions is McFadden (1978).

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