Impact Project

Impact Research Centre, The University of Melbourne,
153 Barry Street, Carlton, Victoria 3053 Australia.

Telephone: (03) 344 7417 (*from overseas:* 61 3 344 7417)
Telex: AA 35185 UNIMEL Telegrams: UNIMELB, Parkville
Facsimile: (03) 344 5104 (*from overseas:* 61 3 344 5104)

Investment with Foresight in General Equilibrium

by

Peter J. Wilcoxen

Harvard University
and
University of Melbourne

Preliminary Working Paper No. IP-35

Melbourne November 1987

ISSN 0813 7986 ISBN 0 642 10232 5

Investment with Foresight in General Equilibrium*

by

Peter J. Wilcoxen
Harvard University and the University of Melbourne

The author would like to thank Alan Powell for his encouragement and many helpful comments, and the University of Melbourne for its financial support while this research was being done.

---

* Original and revised versions received and circulated in January and September 1987, respectively. First issued as an Impact Project paper in November 1987.

Abstract

This paper establishes a link between two important areas of research: applied general equilibrium analysis and the study of adjustment cost models of investment behavior. A hybrid model incorporating both of these features is developed and used to demonstrate a number of interesting interactions between forward-looking investment behavior and general equilibrium. In particular, general equilibrium effects (such as changes in the prices or wages that firms take as given) can alter the amplitude of foresight-related variables by as much as fifty percent. On the other hand, introducing foresight into general equilibrium models can provide a rigorous basis for modeling savings and investment, and it also increases the variety of experiments that can be analyzed. Finally, several numerical methods are described by which the computational requirements of such a model can be reduced to very moderate levels.

Table of Contents

List of Tables

List of Figures

Investment with Foresight in General Equilibrium

by

Peter J. Wilcoxen

1. Introduction

This paper demonstrates the importance of integrating two areas of current research: applied general equilibrium analysis and forward-looking adjustment cost models of investment behavior. Both fields have something to gain from this. General equilibrium models typically have aggregate investment driven entirely by savings, and sometimes also have perfectly malleable capital stocks. Introducing sector specific capital created through forward-looking investment behavior would improve the ability of these models to capture interindustry effects. Furthermore, it would extend the range of policies which could be analyzed to include those which are announced in advance, or are expected to be temporary. On the other hand, the use of general equilibrium allows investment models to incorporate perfect foresight in a natural and intuitive way.

The model presented below is composed of two parts: a small general equilibrium model, and an adjustment-cost investment model. The general equilibrium model has two capital goods, five sectors of production, a representative consumer and the government. Its structure is somewhat similar to the ORANI model [see Dixon, *et al.*, (1982)], although with considerably less detail. The investment model incorporates adjustment costs in the spirit of Eisner and Strotz (1963), Lucas (1967), Gould (1968) and Treadway (1969). In this formulation, investment will depend on Tobin's marginal q [Tobin (1969)], although in the finished model q

has been eliminated (for convenience only) by algebraic substitution.

The resulting model is particularly well suited to analysis of policies that are anticipated with perfect foresight, but it can also be used to study certain forms of incorrect expectations. Results obtained from a wide variety of policy experiments suggest that (1) general equilibrium effects have a strong impact on investment behavior, and (2) changes in investment caused by anticipation of future policies have a significant effect on general equilibrium variables, even in the short run. These observations emphasize the importance of linking the two areas of analysis.

2. An Overview of the Model

The complete model consists of a sequence of short run general equilibrium models linked together by an adjustment-cost investment model. All of the general equilibrium models have the same structure, but each represents the economy at a different point in time. This arrangement can be pictured as a string of beads, with the investment model as the string and the general equilibrium models as the beads. It is useful to discuss briefly the properties of the two basic components.

Each general equilibrium model has the following features. There are five sectors of production, denoted A, B, 1, 2, and K. Sectors A, 1 and 2 produce consumption goods, sector B produces only capital services, and sector K produces a good used in investment. There are two types of capital, K_A and K_B. K_A is created by the investment (I_A) of industry A and is used solely in the production of good A. K_B is created by sector B's investment (I_B) and is rented to sectors 1, 2 and K. All four of the variables K_A, K_B, I_A, and I_B are given exogenously *with respect to the* *general equilibrium model*, although they will be determined endogenously in the investment submodel. Sectors 1, 2 and K are traditional general equilibrium industries which use a perfectly malleable capital stock (K_B) and don't do any investment. The attributes of the sectors are summarized in Table 2.1.

Table 2.1: Characteristics of the Sectors

Sector	Invests In	Capital Used	Output Produced
A	K_A	K_A	Consumption good A
B	K_B	–	K_B capital services
1	–	K_B	Consumption good 1
2	–	K_B	Consumption good 2
K	–	K_B	Raw capital goods

The motivation for this arrangement of sectors is as follows. Sector A is fully integrated into both the general equilibrium and investment models. It uses industry specific capital and must solve both the short run production decision and the long run investment problem. Sector B demonstrates a technique which can be used to reduce the computational requirements of highly disaggregated models while maintaining a consistent investment framework: it does forward-looking investment and rents out its capital stock to all sectors whose investment problem is not explicitly modeled, in this case, sectors 1, 2 and K. This means that all investment in the model will be be forward looking, but without the necessity of having an individual investment model for each sector. Finally, sectors 1 and 2 are traditional zero-profit industries and differ only in capital intensity. Sector K is also a zero-profit industry, but it makes raw capital goods. It was separated from the other sectors to permit experiments involving the price of raw capital goods.

The investment model will be discussed in detail later, but roughly speaking it generates capital stock paths given paths of certain other variables, such as the price of each firm's output. The general equilibrium models were linked to the investment model by a mechanism that allowed expectations to be manipulated. In all, eleven general equilibrium models were used. The first corresponded to the present (period 0), while the last (period 10) was 100 years in the future. Put in terms of the beaded string analogy, there are eleven beads: one at each end and nine in the interior. For example, the equilibria could be assigned to years {0,10,20, · · · 100}, although the interior equilibria need not be evenly spaced in time (this will be discussed in detail in Section 7). Finally, the investment model and the set of general equilibrium models were solved simultaneously, producing a perfect foresight path of the economy.

3. The Investment Model

The firm chooses its investment path to maximize the present value of its dividend stream. Assuming separability of the firm's short and long run optimizations, the results of its short run decision can be summarized by its short run profit function $E(K,V)$, where K is its capital stock and V is a vector of short run variables such as the price of the firm's output. All investment is internally financed, so dividends are short run profits less investment expenditure. Writing $C(I,V)$ for the investment expenditure function, where I is investment, allows the firm's problem to be given by:

$$\max_{I(\cdot)} \int_t^{\infty} (\, E(K,V) - C(I,V)\,)(1-T^D)e^{-rs}\, ds \quad ,$$

$$\text{subject to } \dot{K} = I - \delta K \quad ,$$

where r is the interest rate, δ is the rate of depreciation, and T^D is the dividend tax rate. Given specific short run profit and investment cost functions, the principles of optimal control may be applied to generate first-order conditions for value maximization. These may be combined into a single second-order differential equation that describes the course of the firm's capital stock over time. Finding explicit solutions to this equation for arbitrary policy experiments requires numerical methods; the techniques used here are discussed in detail in Wilcoxen (1985a,b). The investment model consists of two separate value maximization problems, one each for firms A and B, and these are described below.

3.1. Investment by Sector A: Firm Specific Capital

To formulate firm A's investment problem it is first necessary to derive its short run profit function from its production function. Production of good A is taken to be a Cobb-Douglas function of labor and capital, and the firm takes prices as given, so from the production function:

$$X_A = (L_A^P)^{\epsilon_A}(K_A)^{1-\epsilon_A} \quad ,$$

the short run profit function can be shown to be:

$$E_A(K_A,V) = \left(\frac{1-\epsilon_A}{\epsilon_A}\right)\left(\frac{\epsilon_A P_A}{W}\right)^{\frac{1}{1-\epsilon_A}} W K_A \quad ,$$

where L_A^P is the labor used in production by industry A, and W is the wage rate. Notice that the firm must form expectations about the future course of wages and the price of its product in order to be able to compute the earnings of its future capital stock.

In addition to the short run profit function, the firm's investment cost function is required to obtain a solution to the optimization problem. This function could be specified directly, but in the current model it was instead derived from a particular choice of the firm's investment-good production function. To be specific, the firm produces its own investment good by purchasing raw capital (X_K^A) and hiring labor (L_A^I) to install it. The amount of labor required is proportional to the square of the amount of raw capital. This description can be summarized by the Leontief production function below:

$$I_A = \min\left\{ X_K^A \, , \, \left(\frac{L_A^I}{\theta_A} \right)^{\frac{1}{2}} \right\} \quad ,$$

where X_K^A is raw capital, L_A^I is labor used by industry A in the construction of its investment good, and θ_A is a parameter. The corresponding cost function is:

$$C_A(X_K^A, V) = (P_K X_K^A + W L_A^I)(1 - T^S) \quad ,$$

where T^S is an investment subsidy. Minimizing investment costs given the production function above requires that the following hold:

$$I_A = X_K^A = \left(\frac{L_A^I}{\theta_A}\right)^{1/2} \quad .$$

Solving for X_K^A and L_A^I in terms of I_A gives:

$$X_K^A = I_A \quad ,$$

and

$$L_A^I = \theta_A I_A^2 \quad .$$

Finally, this means the investment cost function can be written as shown:

$$C_A(I_A, V) = (P_K I_A + W\theta_A I_A^2)(1 - T^S) \quad .$$

Two important remarks must be made about this equation. First, because θ_A is not zero, the firm faces internal costs of adjustment -- the cost of new capital is convex in investment. Second, adjustment costs depend on gross rather than net investment. In the steady state, gross investment will be equal to depreciation, so steady state adjustment costs depend on the size of the capital stock. This feature will be relevant for a simulation presented in Section 9, but it could easily be removed by rewriting the problem in terms of net investment.

Returning to development of the investment model, finding the path of the capital stock requires solving an optimal control problem using the short run profit and investment cost functions above. The result is a system of differential equations -- the problem's first-order conditions -- which must be solved to produce an explicit expression for the capital stock over time. For sector A, these first-order conditions are:

$$\lambda_A = (P_K + 2W\theta_A I_A)(1-T^D)(1-T^S) \quad ,$$

$$\dot{\lambda}_A = (r+\delta)\lambda_A - \beta(1-T^D) \quad ,$$

$$\dot{K} = I - \delta K \quad ,$$

where

$$\beta = \left(\frac{1-\epsilon_A}{\epsilon_A}\right)\left(\frac{\epsilon_A P_A}{W}\right)^{\frac{1}{1-\epsilon_A}} W \quad .$$

These may be combined to form the single second-order differential equation given below:

$$\ddot{K}_A - \dot{K}_A\left(r - g - \frac{\dot{W}}{W}\right) - \delta K_A\left(r + \delta - g - \frac{\dot{W}}{W}\right) =$$

$$\frac{P_K}{2W\theta_A}\left((r+\delta-g) - \frac{\dot{P}_K}{P_K} - \frac{\beta}{P_K(1-T^S)}\right) \quad ,$$

where g is given by the function:

$$g = -\frac{\dot{T}^S}{(1-T^S)} - \frac{\dot{T}^D}{(1-T^D)} \quad .$$

(g can be interpreted as the rate of growth of the cost—in terms of dividends—of a dollar of investment.) The second order equation above fully describes the optimal path of the capital stock for given paths of P_A, P_K, W, T^D and T^S.

For most experiments, it will be impossible to find a general solution analytically, so numerical methods must be used. The technique used here is called finite differences, and it is discussed in detail in Wilcoxen (1985a). Specific aspects of the method relevant for the current model are presented briefly in Section 7 below.

3.2. Investment by Sector B: General Purpose Capital

The other investment sector, industry B, produces only capital services. It takes prices as given, so its short run profit is determined solely by its capital stock and the corresponding capital rental price, as shown below:

$$E_B(K_B, V) = \rho K_B \quad ,$$

where ρ is the rental price of a unit of general purpose capital. The sector's investment cost function is similar to that of the previous industry and is given by:

$$C_B(I_B, V) = (P_K I_B + W \theta_B I_B^2)(1 - T^S) \quad .$$

The first-order conditions for this problem are given below:

$$\lambda_B = (P_K + 2W\theta_B I_B)(1-T^D)(1-T^S) \quad ,$$

$$\dot{\lambda}_B = (r+\delta)\lambda_B - \rho(1-T^D) \quad ,$$

$$\dot{K}_B = I - \delta K \quad .$$

As before, these may be combined into a single second-order differential equation:

$$\ddot{K}_B - \dot{K}_B\left(r - g - \frac{\dot{W}}{W}\right) - \delta K_B\left(r + \delta - g - \frac{\dot{W}}{W}\right) =$$

$$\frac{P_K}{2W\theta_B}\left((r+\delta-g) - \frac{\dot{P}_K}{P_K} - \frac{\rho}{P_K(1-T^S)}\right) \quad ,$$

where g is defined as above.

4. The Short-Run General Equilibrium Model

The general equilibrium model includes the two investment sectors (A,B), three "traditional" industries (1,2,K), one consumer and the government. The traditional industries rent capital from sector B and earn no short run profits. Consumption goods are produced by industry A and by traditional sectors 1 and 2. The third traditional sector, K, produces raw capital goods used in investment. All prices in the model are those received by producers, except for that of raw capital goods which is the purchaser's price.

4.1. Investment Sectors

As discussed above, production in sector A is a Cobb-Douglas function of labor used in production and the industry's capital stock:

$$X_A = (L_A^P)^{\epsilon_A}(K_A)^{1-\epsilon_A} \quad .$$

Maximizing profits on existing capital implies the labor demand equation shown below:

$$L_A^P = \left(\frac{\epsilon_A P_A}{W}\right)^{\frac{1}{1-\epsilon_A}} K_A \quad .$$

Investment is obtained from the optimal capital stock trajectory according to the following expression:

$$I_A = \dot{K}_A + \delta K_A \quad .$$

A finite difference approximation is used for $\dot{K}$, so investment is completely determined by the optimal path of capital. Cost minimization in production of investment goods generates the demands shown below for raw capital and investment labor:

$$X_K^A = I_A \quad ,$$

$$L_A^I = \theta_A I_A^2 \quad .$$

Finally, revenue less wage costs in production less investment costs gives pre-tax dividends:

$$D_A = P_A X_A - WL_A^P - \left(P_K X_K^A + WL_A^I\right)(1-T^S) \quad .$$

Industry B does not produce any goods, so its behavior is entirely determined by the optimal path of its capital stock. As with industry A, investment is obtained from the firm's capital accumulation constraint:

$$I_B = \dot{K}_B + \delta K_B \quad .$$

Deriving the demands for raw capital and investment labor produces the equations shown below:

$$X_K^B = I_B \quad ,$$

$$L_B = \theta_B I_B^2 \quad .$$

Gross dividends are simply revenue less investment costs:

$$D_B = \rho K_B - (P_K X_K^B + WL_B)(1-T^S) \quad .$$

The equations above fully describe the short run behavior of the special investment sectors in the model.

4.2. Other Production

Three other sectors are included in the model: industries 1, 2 and K. These sectors rent their capital from industry B at price ρ and do not invest. Production in each sector is Cobb-Douglas, as shown below, where $i \in \{1,2,K\}$:

$$X_i = \gamma_i (L_i)^{\epsilon_i} (K_B^i)^{1-\epsilon_i} \quad .$$

Straightforward optimization generates the factor demand equations shown below:

$$L_i = \frac{1}{\gamma_i} X_i \left(\frac{\rho \epsilon_i}{W(1-\epsilon_i)} \right)^{1-\epsilon_i} \quad ,$$

$$K_B^i = \frac{1}{\gamma_i} X_i \left(\frac{W(1-\epsilon_i)}{\rho \epsilon_i} \right)^{\epsilon_i} \quad .$$

Finally, each sector is constrained to earn zero pure profits. For industries 1 and 2 this condition is:

$$X_i P_i = W L_i + \rho K_B^i \quad .$$

Because the price of raw capital goods is the purchaser's cost, the zero pure profit condition for industry K is slightly different, as shown below:

$$X_K P_K = (1 + T_S^K)(W L_K + \rho K_B^K) \quad ,$$

where T_S^K is the sales tax on capital goods.

4.3. The Consumer

The single consumer in the model supplies labor and owns both of the investment firms, so income includes wages, dividends and transfers. All income in each period is spent on consumption; there is no saving (except earnings retained by firms) and the consumer does not forecast future earnings. Thus, the consumer's budget constraint is the following:

$$C = WL(1-T^W) + (D_A+D_B)(1-T^D) + TR \quad ,$$

where C is consumption expenditure, and T^W is the tax on wages. Labor supply is inelastic, so the demand system below can be derived from simple utility maximization:

$$X_A^C P_A(1+T_S^A) = \alpha_C^A C \quad ,$$

$$X_1^C P_1(1+T_S^1) = \alpha_C^1 C \quad ,$$

$$X_2^C P_2(1+T_S^2) = \alpha_C^2 C \quad ,$$

where the α's are Cobb-Douglas exponents and T_S^A, T_S^1, and T_S^2 are sales taxes on goods A, 1 and 2 respectively.

4.4. The Government

The government is constrained to balance its budget, so spending is equal to tax revenue less transfers and subsidies. Revenue is raised by dividend taxes, wage taxes and sales taxes, while transfers are made to the consumer and subsidies are paid on investment expenditure. Thus, the government's budget is given by the following equation:

$$
\begin{aligned}
G = T^D(D_A + D_B) - T^S(P_K(X_K^A + X_K^B) + W(\theta_A I_A^2 + \theta_B I_B^2)) \\
+ T_S^A P_A X_A + T_S^1 P_1 X_1 + T_S^2 X_2 P_2 + T_S^K P_K X_K \\
+ T^W WL - TR \quad .
\end{aligned}
$$

The government demand system is derived from a Cobb-Douglas utility function and consists of the following equations:

$$
\begin{aligned}
X_A^G P_A (1 + T_S^A) &= \alpha_G^A G \quad , \\
X_1^G P_1 (1 + T_S^1) &= \alpha_G^1 G \quad , \\
X_2^G P_2 (1 + T_S^2) &= \alpha_G^2 G \quad .
\end{aligned}
$$

4.5. Market Clearing

The final group of equations necessary to define the model is the set of market clearing conditions. For goods A, 1 and 2, total demand is the sum of private and government demand. Demand for good K is the sum of raw capital demand by the two investment sectors. The four equations are:

$$X_A = X_A^C + X_A^G \quad ,$$

$$X_1 = X_1^C + X_1^G \quad ,$$

$$X_2 = X_2^C + X_2^G \quad ,$$

$$X_K = X_K^A + X_K^B \quad .$$

In addition, factor market clearing for labor and capital B requires the following:

$$L = L_A^P + L_A^I + L_B + L_1 + L_2 + L_K \quad ,$$

$$K_B = K_B^1 + K_B^2 + K_B^K \quad .$$

4.6. Other Equations

In addition to all of the equations above, a price deflator was also incorporated into the model. The index, ζ, was defined as the cost of the current $C+G$ bundle at current prices divided by its cost at period zero's initial prices:

$$\zeta = \frac{X_A P_A T_S^A + X_1 P_1 T_S^1 + X_2 P_2 T_S^2}{X_A (P_A T_S^A)_0 + X_1 (P_1 T_S^1)_0 + X_2 (P_2 T_S^2)_0} \quad ,$$

where the variables in enclosed in parentheses are initial values for period zero. Note that in many experiments period zero prices will change, so the deflator may move away from one at time zero.

5. Expectation Formation

The model also included a special set of variables to allow manipulation of the expectations driving the investment model. Inspection of the investment problems outlined in Section 3 reveals that only the variables ρ, W, P_A, P_K, T^D and T^S are relevant for the two optimizations. For each of these variables an expectation was formed by combining its true general equilibrium value with an exogenous component. For example, the wage used in the investment submodel, W^e, was formed out of the true general equilibrium wage, W, and a fixed expectation W^z, as shown below:

$$W^e = (W)^{\lambda_N}(W^z)^{1-\lambda_N} \quad ,$$

where λ_N was a parameter ranging from zero to one. When $\lambda_N=1$, firms have perfect foresight; when $\lambda_N=0$, the expected wage is set to the exogenous value W^z. This procedure was also carried out for expectations of exogenous variables (T^D and T^S), but a separate parameter, λ_X, was used.

These two parameters allow simulations to be run under different assumptions about the extent to which firms can predict future variables. When both λ_N and λ_X are set to 1, firms have perfect foresight. If $\lambda_N=0$ and $\lambda_X=1$, firms perfectly anticipate changes in tax rates but do not correctly predict general equilibrium consequences. This specification was used to examine how much of the period zero effects of a policy were due to firms' ability to forecast the policy's full general equilibrium impact.

6. The Data Set

A trial data set constructed to allow testing of the model is presented in Appendix 1. It has a number of interesting features, but does not represent any particular economy. One of its most important characteristics is that the patterns of private consumption and government spending are identical, so no composition effects arise when changes in taxes induce transfers of income between the private and government sectors. This feature allows government spending to be made exogenous and transfer payments made endogenous, so overall gains or losses can be measured directly from changes in consumption.

7. Solving the Model

The complete model consists of an intertemporal investment module linked to eleven single-period general equilibrium models. Solving the investment problem requires solving a system of differential equations; solving any one of the general equilibrium models requires solving a large system of nonlinear equations. Both components of the model must be solved simultaneously, so the entire process is not trivial. This section will set out the basic approach used and discuss how close the numerical solution will be to the true solution.

7.1. Finite Differences

In order to solve the investment problems of firms in sectors A and B, the second-order differential equations describing the solutions were converted to their finite difference equivalents. This method, which is described in detail in Wilcoxen (1985a), essentially replaces a differential equation with a system of Taylor series expansions which hold at successive points in time. The solution is then found by solving the equations simultaneously. This technique is numerically stable even when exponential terms appear in the solution, as in most economic problems.

To illustrate how the finite difference method works, consider the differential equation shown below:

$$a(t)\ddot{K}(t) + b(t)\dot{K}(t) + c(t)K(t) = d(t) \quad .$$

Replacing all of the differentials with their finite difference equivalents produces the following approximation:

$$a\left(\frac{K(t+\Delta t)-2K(t)+K(t-\Delta t)}{(\Delta t)^2}\right)+b\left(\frac{K(t+\Delta t)-K(t-\Delta t)}{2\Delta t}\right)+cK(t)=d \quad ,$$

where coefficient functions a,b,c and d are evaluated at t. The complete model consists of a set of these equations, each holding at a successive point in time.

Notice that the equation above is linear in capital, so if the time paths of the coefficients are known, the system may be solved easily using Gaussian elimination. In a general equilibrium analysis, however, these coefficients will be nonlinear functions of variables (such as the price of the firm's output) which are not known prior to solving the investment problem.

7.2. Johansen Linearization

To solve the general equilibrium model, the method of Johansen was employed: the model is converted to its linearized percentage change form and solved by Gaussian elimination. (Roughly speaking, percentage change form is a first-order Taylor expansion.) The advantages of this over iterative techniques such as Newton's method or Scarf's algorithm are computational speed and flexibility of use: solving the model is fast because elimination is used rather than iteration, and the method is flexible because the division of variables into endogenous and exogenous sets can be changed at will. Johansen's method is described in Dixon, *et al.* (1982).

Linearization of the general equilibrium model requires that the finite difference investment module also be linearized for compatibility. The complete model thus involves a Taylor series expansion in both time (finite differences) and variables (Johansen linearization). Although the solution is an approximation and is subject to two sources of truncation error, it may be made arbitrarily close to the true result by decreasing the step size used in time and variables; this topic will be discussed in detail in Section 7.4.

As a brief illustration of Johansen linearization, consider sector A's labor demand equation, repeated below:

$$L_A^P = \left(\frac{\epsilon_A P_A}{W} \right)^{\frac{1}{1-\epsilon_A}} K_A \quad .$$

Converting this to percentage change form produces the equation shown below, where lowercase letters are used for variables expressed as percentage changes:

$$l_A^P = k_A + \phi(p_A - w) \quad ,$$

where

$$\phi = \frac{1}{1-\epsilon_A} \quad .$$

This equation gives the percentage change in the demand for labor in industry A as a function of the percentage changes in the industry's capital stock, output price and the wage. The coefficient ϕ is calculated from the data set.

Linearization of the model produces a system of equations and percentage change variables for which the total number of variables exceeds the number of equations by the number of exogenous variables. Denoting the coefficients of the linearized system as array F and the variables as vector X, the model can be written as shown:

$$\mathrm{FX} = 0 \quad .$$

Partitioning X into endogenous and exogenous vectors U and V allows the model to be written:

$$\mathrm{F}^1\mathrm{U} + \mathrm{F}^2\mathrm{V} = 0 \quad ,$$

where the columns of F have been partitioned into arrays F^1 and F^2 according to the mapping of X into U and V. Finally, this equation can be rewritten as shown:

$$F^1U = -F^2V \quad .$$

Since F^1 is square and V is known, the vector of endogenous variables U can be found by Gaussian elimination.

One of the most important advantages of Johansen technique over other methods is that it is easy to tailor the partition of variables to suit a particular experiment. For example, if the dividend tax rate is to be increased, the government could return the extra revenue as lump-sum transfers, or it could simply spend it. With the linearized model described above, either assumption could be employed by making the appropriate variable exogenous and setting its percentage change to zero. For the first specification, the government's budget would be exogenized at zero percentage change and the level of transfers endogenized; for the second, transfers would be exogenized and government spending made endogenous.

7.3. Testing the Complete Model

Once the two finite difference investment modules were combined with the eleven general equilibrium models and the entire system was linearized, a number of special experiments were run to check that the model was programmed correctly. Three such simulations were used: (1) a homogeneity test, (2) a surprise dividend tax at time zero, and (3) an increase in the tax on wages.

The first test consists of a simultaneous increase in the price deflator and nominal transfer payments. The effect of this should be to raise all nominal variables by the amount of the increase, and to leave all real variables unchanged. This result was obtained, indicating that the model was free of gross programming errors.

The second test, increasing the dividend tax at time zero, is an unavoidable pure profits tax. As such, it should have no effect on the capital stock or output of any industry, although dividends and firm values should fall by the amount of the tax. There will be a large shift of income from consumers to the government, but since both sectors have the same patterns of demand, there will be no composition effects. These results were correctly generated by the model.

The final test run was an increase in the wage tax paid by consumers. Since the labor supply is inelastic, the effect of this should be a simple transfer of income to the government. This result was also generated by the model.

Together these experiments provide strong (albeit indirect) evidence that the model was free of programming errors. Having verified this, it was necessary to check that the linearized model would correctly converge on less trivial experiments.

7.4. Verifying Convergence

In essence the model is a system of partial differential equations in time and variables, and solving it is accomplished by integrating over time using finite differences and over variables using Euler's method. The accuracy of the solution depends on the step size used in these integrations: the solution will approach the true solution as the step size in both time and variables is made infinitesimal. To verify that the model was formulated correctly, it was necessary to show that the numerical solution could be made arbitrarily close to the true (analytical) solution for a particular experiment. Furthermore, it was important to demonstrate that reasonably accurate results could be obtained with a fairly small number of steps in each dimension.

At this point it is useful to discuss how, in practice, the two step sizes can be decreased. In terms of the finite difference approximation, decreasing the step size requires increasing the number of equations in the finite difference approximation. Roughly speaking, doubling the number of points (and hence equations) should reduce truncation error by a factor of two. Extensive discussion of this can be found in Wilcoxen (1985a). On the other hand, accuracy of the Johansen solution can be improved by using Euler's method. In this approach, the overall shock is divided into a number of steps, the model is solved for the first step, the resulting equilibrium is then subjected to the second step, etc. The numerical solution can be made arbitrarily close to the true solution by increasing the number of steps (see Dixon, *et al.* (1982)).

To investigate the practicality of obtaining accurate solutions at reasonable step size, the investment modules were linearized and solved for a typical experiment. (The general equilibrium variables were set exogenously.) The results were compared with those from a finite difference system which had not been linearized, and with the analytical solution.

The experiment chosen was an increase in the dividend tax rate from 10% to 20%, to take effect ten years in the future. This experiment is particularly useful because it can be solved analytically, so it is possible to see how close the numerical solution is to the truth. To summarize the experiment briefly, the announcement of the tax causes firms to pay large dividends immediately before the tax takes effect. This drives down investment, so when the tax is implemented the capital stock will be lower than it would have been. Once implemented, however, the tax falls on pure profits, so firms return to their pre-announcement behavior and the economy gradually returns to the original steady state capital stock, although owners of capital have suffered a windfall loss (see Wilcoxen (1985a,b)). To compare the accuracy of different solutions, attention was focused on the value obtained for the capital stock in period ten (this turns out to be the least accurate point of a given solution). Table 7.4.1 shows the effect on the solution of increasing the density of finite difference approximations (grid density) and the effect of decreasing the exogenous shock step size (listed as increasing iterations).

Table 7.4.1: The Effect of Grid Density and Iterations on the Value of the Capital Stock at Period 10†

	Number of Grid Points			
Iteration	9	19	39	79
1	.9623	.9376	.9247	.9180
2	.9605	.9361	.9236	.9174
4	.9596	.9353	.9323	.9171
8	.9591	.9350	.9229	.9169
∞	.9586	.9346	.9227	.9168

† The true value of $K(10)$ is .9113.

Several things are readily apparent. First, increasing either grid density or iterations improves the solution as expected. Because of the Taylor linearizations in each dimension, the difference between values obtained from successive halvings of the step size should decrease by roughly a factor of two, which corresponds to what is shown in the table. Second, from the initial 1-iteration, 9-grid-point solution, accuracy increases most rapidly by increasing the number of finite difference grid points. This indicates that the error introduced by the finite difference approximation overwhelms that of the Johansen linearization. In fact, doubling the grid density improves the solution by more than increasing the number of Johansen iterations arbitrarily. (The values for infinite iterations are obtained from finite difference solutions to models which were not linearized). Johansen linearization error is trivial, but that introduced by finite differences is not.

The importance of grid spacing cannot be understated: it is crucial to the numerical accuracy of the solution. Unfortunately, doubling the number of grid points increases the computational load of the problem by more than a factor of four since the number of operations required for Gaussian elimination rises more rapidly than the square of the number of equations. This means that the binding computational constraint will be accuracy of the finite difference equations.

Fortunately, it is possible to improve the finite difference approximation by a method which does not require additional grid points. Knowledge of the path of the exogenous shock can be exploited to allow the spacing of grid points to be chosen strategically. Roughly speaking, the truncation error in a Taylor expansion is the product of a high order derivative and the distance between grid points. Error may be diminished simply by placing points close together where the derivative is large, at the expense of spreading them more thinly where the derivative is small. In practice this involves moving points from large times, say 80 or 90 years in the future, to times near implementation of the policy. Table 7.4.2 shows five possible allocations of nine points to times between 0 and 100 years (eleven points are shown because the end points are included).

Table 7.4.2: A Selection of Grid Spacings

	Grid				
Point	A	B	C	D	G
0	0	0	0	0	0
1	10	5	5	5	5
2	20	10	7	7	7
3	30	20	10	9	9
4	40	30	20	10	10
5	50	40	30	20	15
6	60	50	40	30	20
7	70	60	50	40	35
8	80	70	60	50	50
9	90	80	70	60	75
10	100	100	100	100	100

For an experiment like the dividend tax implemented at year 10, the system will almost be back to the steady state at late years like 90, so it would be desirable to move that point to an earlier time. The result of this is shown in grid B above, where the point at 90 has been moved to year 5. Continuing this rearrangement produces the set of grids shown in the table. These grids were then used to solve the dividend tax experiment, with the results shown in the table below.

Table 7.4.3: The Effect of Grid Choice and Iterations on the Value of the Capital Stock at Period 10†

	Grid				
Iterations	A	B	C	D	G
1	.9623	.9407	.9323	.9233	.9199
2	.9605	.9392	.9310	.9226	.9192
4	.9596	.9384	.9303	.9222	.9187
8	.9591	.9381	.9300	.9219	.9185
∞	.9586	.9377	.9297	.9217	.9183

† The true value of $K(10)$ is .9113.

The important feature shown by the table is that rearrangement of a limited number of grid points can produce a solution almost as accurate as increasing the density of a uniform grid by a factor of eight. This result is vital to the feasibility of obtaining accurate numerical solutions at reasonable cost. In all of the experiments described below, the last grid, G, was used to minimize finite difference truncation error. Note that convergence of the solution across exogenous shock iterations is obtained with any grid structure. (That is, Johansen convergence is not disturbed by choice of grid structure.)

8. Partitioning

As discussed above, the flexibility of the partition of variables into endogenous and exogenous sets is a particular strength of the Johansen technique. Table 8.1 shows the basic list of exogenous variables used in the experiments discussed here.

Table 8.1: Exogenous Variables

Symbol	Description
K_A	Sector A capital (period 0 only)
K_B	Sector B capital (period 0 only)
L	Total labor supply
G	Government spending
T_W	Tax on wages
T_S^A	Sales tax on good A
T_S^1	Sales tax on good 1
T_S^2	Sales tax on good 2
T_S^K	Sales tax on good K
T^D	Dividend tax
T^S	Investment subsidy
γ_1	Technical change parameter, industry 1
γ_2	Technical change parameter, industry 2
γ_K	Technical change parameter, industry K
ρ^z	Exogenous expectations, rental price
W^z	Exogenous expectations, wage rate
P_K^z	Exogenous expectations, raw capital price
P_A^z	Exogenous expectations, price of good A
T^{D*}	Exogenous expectations, dividend tax rate
T^{S*}	Exogenous expectations, investment subsidy
r	Interest rate
ζ	Price deflator

Two consequences of this choice of exogenous variables should be noted. First, since government spending is exogenous and transfers are endogenous, the revenue accruing from any tax change will all be passed back to households as lump-sum transfers. This was done to allow different policies to be compared simply by comparing private consumption. The second important feature is that the price deflator is held constant. This means that all prices in the model will be normalized by the price of an aggregate consumption good.

9. Results

This model is particularly well adapted to the study of interactions between general equilibrium effects and investment. Simulations addressing three topics are discussed below: (1) an assessment of the importance to investment of foresight regarding general equilibrium effects, (2) demonstration of the interaction between general equilibrium experiments and investment under perfect foresight, and (3) capital accumulation in response to an increase in the labor force.

9.1. The Importance of General Equilibrium Effects

One question of some interest is how much difference the accuracy of agents' expectations makes to the solution. In particular, it might be plausible to assume that agents have perfect foresight with respect to the exogenous variables of the model while having fixed beliefs about endogenous general equilibrium variables. To be concrete, a firm might know the future path of the dividend tax accurately from government proclamations, but be unable to assess correctly the full general equilibrium effects of the tax. This issue can be examined by manipulating the parameters λ_N and λ_X.

The experiment chosen for analysis was the announced dividend tax to be implemented in period ten. With $\lambda_X=1$ and $\lambda_N=1$, the model generated a perfect foresight equilibrium in which agents accurately forecast the course of future taxes and were able to predict the full general equilibrium effects of the taxes. The model was also solved with $\lambda_X=1$ and $\lambda_N=0$. In this case, agents expected all endogenous variables to be unchanged. (This will be correct in the long run since the dividend tax is a pure profits tax in the long run.) The results of these two simulations are shown in Figure 9.1.1, in which capital stocks A and B, investment A and B, the wage, the rental price of capital B, the price of good A, dividends A and B and consumption are shown. Note that consumption is unchanged in the long run because the government returns any extra revenue through lump- sum transfers. In the figure, P indicates perfect foresight expectations, while F indicates fixed (exogenous) expectations.

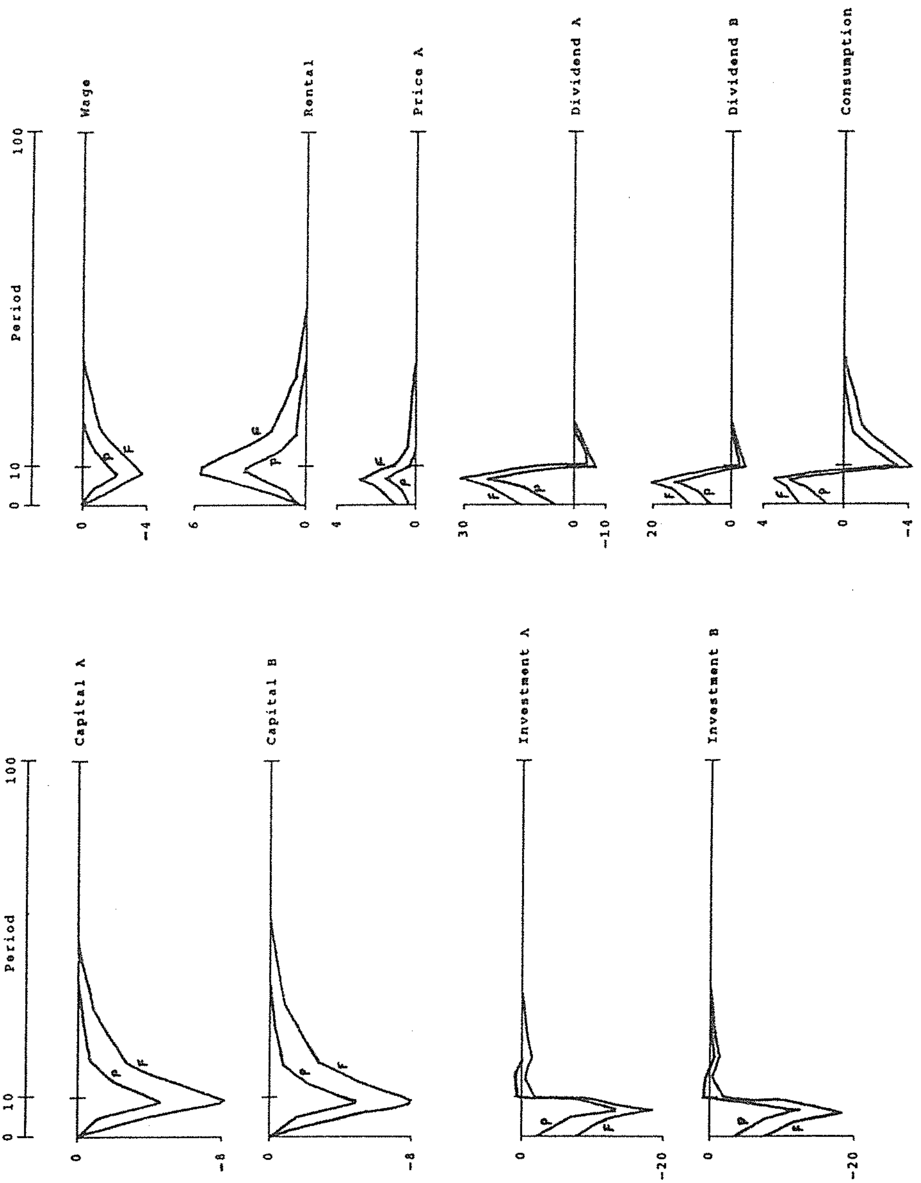
The striking feature of Figure 9.1.1 is that perfect foresight over endogenous variables attenuates the response of the model by about fifty percent. Notice that this occurs even in period zero: investment falls less than half as much under perfect foresight. This occurs because the price rise which accompanies declining capital in sectors A and B raises the amount firms can earn on a given capital stock before implementation of the tax, thus raising the optimal amount of capital in those periods.

9.2. Indirect Dynamic Effects

The model can also be used to investigate the dynamic effects of policies which influence the investment problem only indirectly through general equilibrium effects. One example of this is a change in sales taxes.

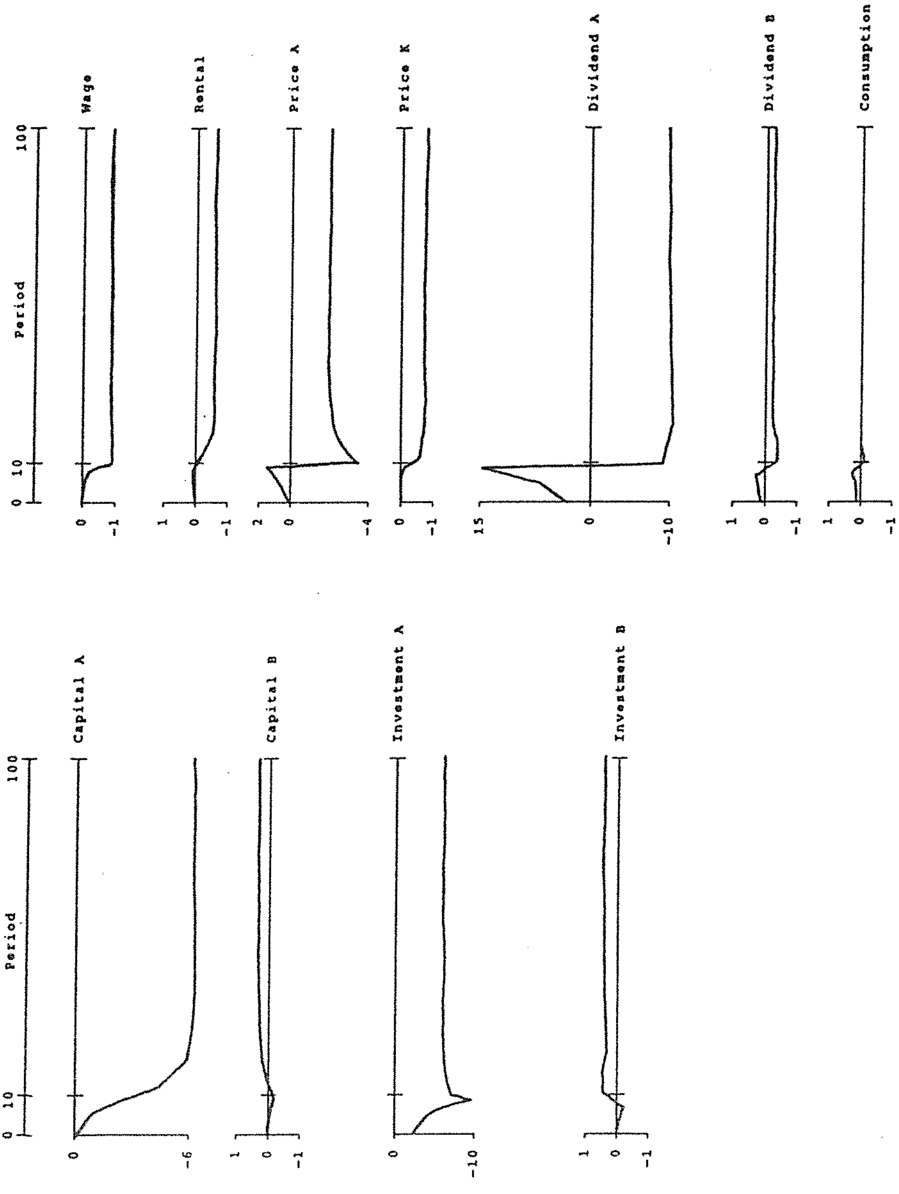
Figure 9.2.1 shows the effect on a number of model variables of an announced increase in the sales tax on good A to be implemented at year ten. As might be expected, the sales tax leads to a drop in the optimal amount of capital in that sector. As shown in the figure, in the long run capital in sector A falls to less than

Figure 9.1.1: Dividend Tax Increase Under Different Expectations Assumptions



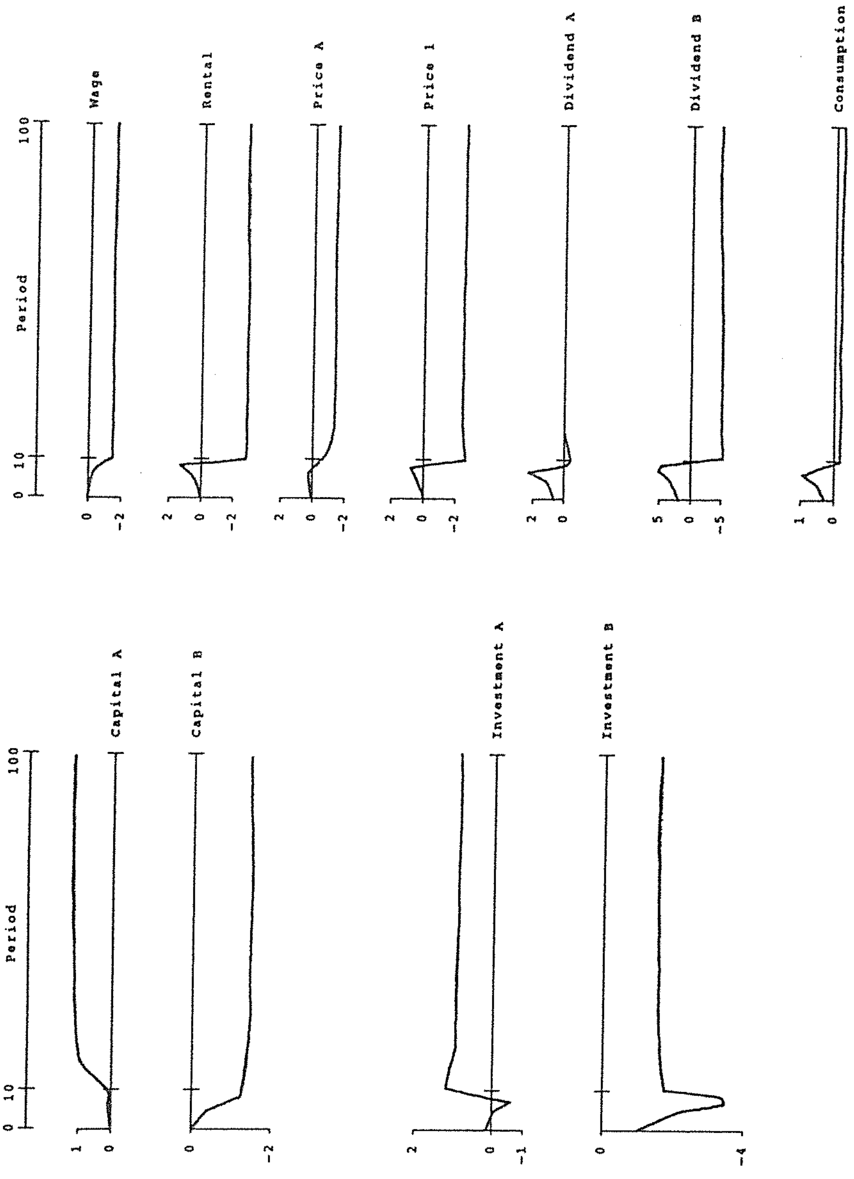
P indicates a perfect foresight path, F a path with fixed (exogenous) expectations.
All graphs show the percentage change in a variable from its original value in each year.

Figure 9.2.1: Sales Tax on Good A



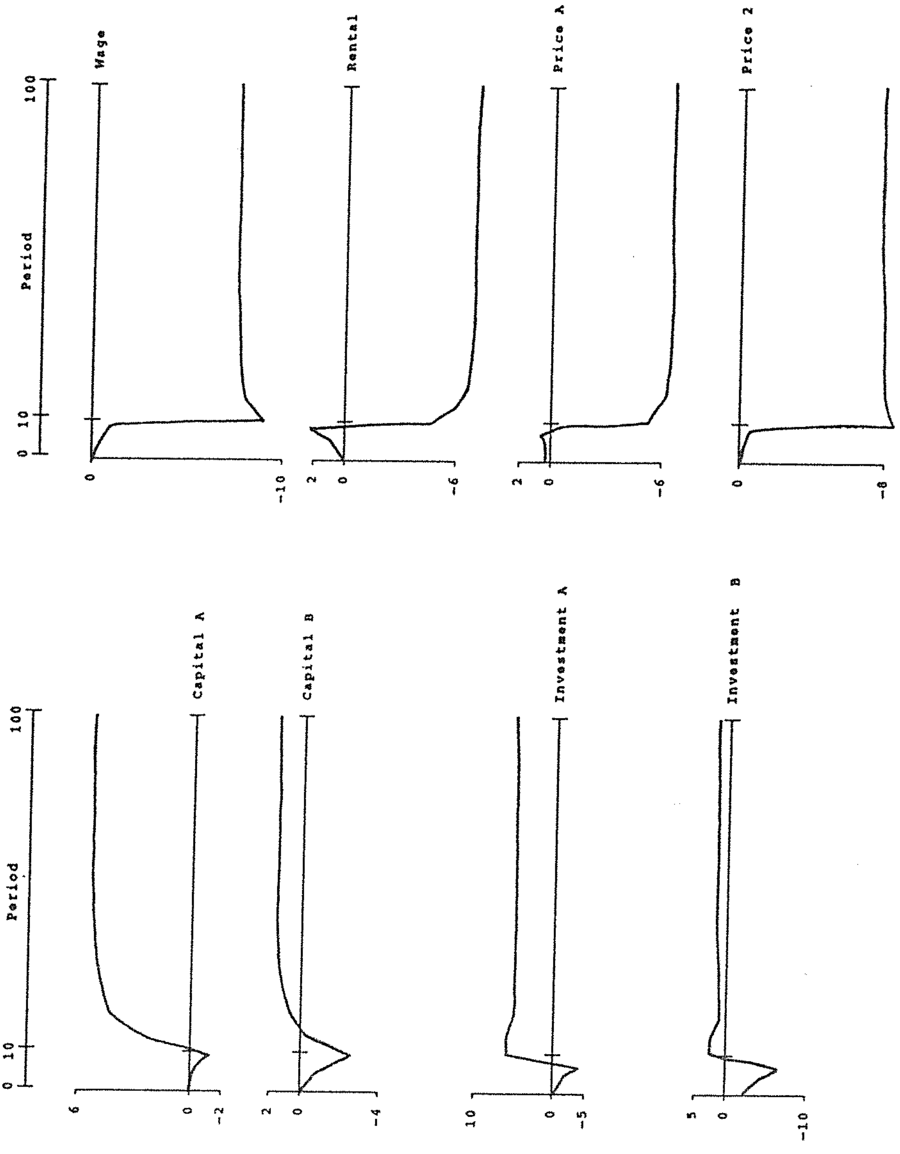
All graphs show the percentage change in a variable from its original value in each year.

Figure 9.2.2: Sales Tax on Good 1



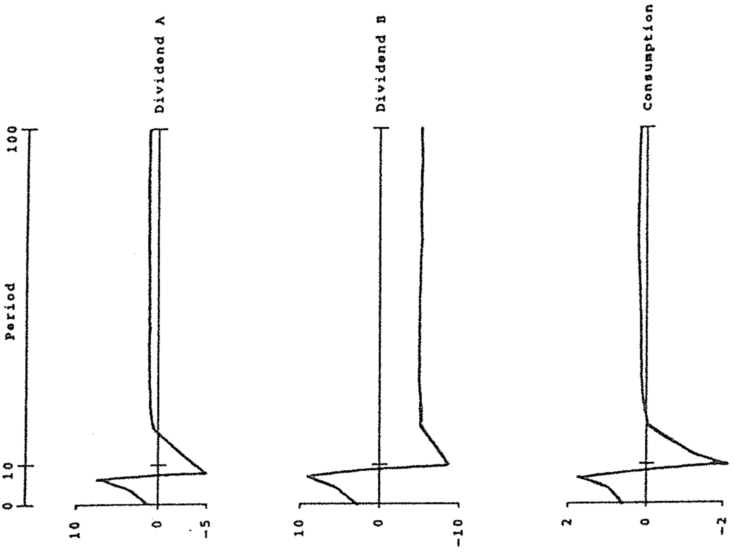
All graphs show the percentage change in a variable from its original value in each year.

Figure 9.2.3: Sales Tax on Good 2



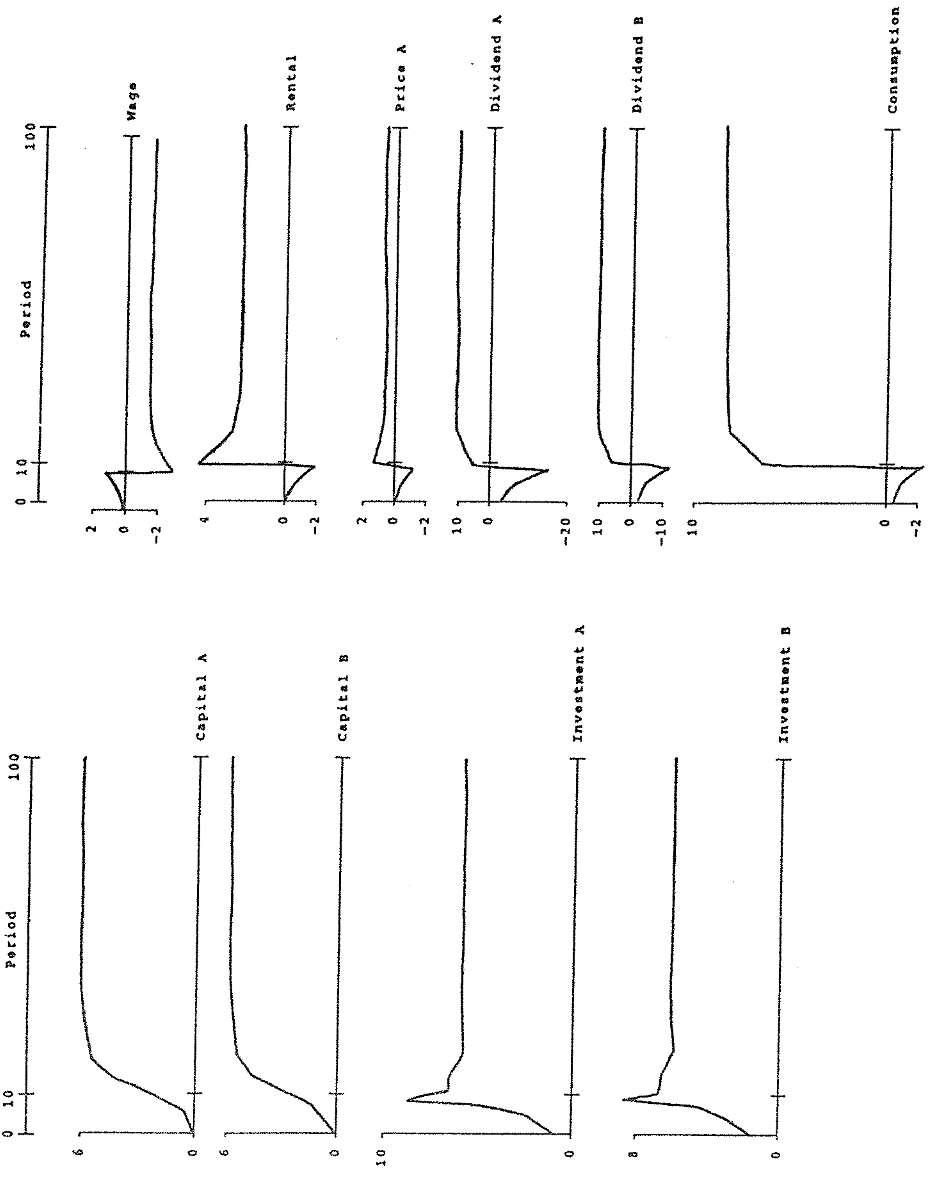
All graphs show the percentage change in a variable from its original value in each year.

Figure 9.2.3, continued: Sales Tax on Good 2



All graphs show the percentage change in a variable from its original value in each year.

Figure 9.3.1: Increase in Labor Supply and Government Spending



All graphs show the percentage change in a variable from its original value in each year.

60% of its initial value. This decline in sector A frees up labor, causing a drop in the wage rate. Lower wages benefit sector B by lowering its investment costs, so investment increases and K_B rises over time. This drives down the rental price of capital B, so producer prices of goods 1, 2 and K must fall since the wage also fell. The producer price of goods A also falls, although the purchaser's price has risen because of the tax. Finally, dividends in sector A fall as expected; dividends in B also fall because of the reduced rental price of capital B.

A sales tax on good 1 produces similar effects, as shown in Figure 9.2.2. Sector 1 is relatively capital intensive, so when it shrinks due to the tax, much capital of type B is freed up. This forces down the rental price of capital, lowers sector B's investment and eventually drives down the stock of capital B. Lower investment and a smaller sector 1 contribute to the decline in wages, which benefits sectors A, 2 and K.

One more sales tax experiment is of some interest: an increase in the tax on good 2. The results of the simulation are shown in Figure 9.2.3. The main difference from the previous two experiments is that the capital stock in both sectors A and B rises over time. The source of this curious result is that sector 2 is very large and very labor intensive. A small contraction in its output leads to a substantial drop in the wage. This lowers investment costs for both sectors A and B, so investment rises and both capital stocks grow. Finally, growth of the capital stocks causes the rental price of capital B and the price of good A to fall. In the end, sector A has gained, producing slightly higher dividends in the long run, while sector B's dividends fall considerably.

9.3. An Increase in Labor Supply

One last experiment, an increase in the labor force, highlights an interesting property of the model. In the long run, an increase in the labor force should produce a corresponding increase in all capital stocks as the economy returns to its initial capital-labor ratio. Also, all prices should return to their original values. This behavior is not observed in this model, as shown in Figure 9.3.1. The experiment is an anticipated permanent 10% increase in the labor force at period ten.

The reason the model produces this behavior is that adjustment costs are a function of gross investment. As the steady state capital stock rises, so does the share of factors devoted to adjustment costs. In fact, these costs rise with the square of capital stock, so the overall economy is subject, in effect, to diminishing returns to scale. For this reason, the capital stocks do not rise by 10%, and neither does consumption. As mentioned above, this feature could be eliminated, if desired, by rewriting the adjustment cost function in terms of net investment.

10. Extensions

A large number of interesting extensions are suggested by the results discussed here. First, as mentioned above, the investment modules could be modified to be functions of net investment. Also, this model limits intertemporal optimization to investment; in particular, lifecycle consumption is not used here. Such features could be added in a manner analogous to the approach used for investment. Finally, numerous improvements could easily be made to either the investment sector specification or the general equilibrium model itself.

11. Conclusions

It is clear from this analysis that integration of general equilibrium models with perfect foresight investment sectors is computationally feasible. Furthermore, the results of the effort are of economic interest as it has been shown that (1) general equilibrium effects are important in the investment problem, and (2) that forward-looking investment behavior has consequences for general equilibrium analysis, even in the very short run, as evidenced by the period zero changes in investment seen above. Further work must be done, but the objective is feasible and valuable.

12. References

Birkhoff, G. and G.C. Rota (1969): *Ordinary Differential Equations,* London: Blaisdell.

Dixon, P.B., et al. (1982): *ORANI: A Multisectoral Model of the Australian Economy,* Amsterdam: North-Holland.

Eisner, R. and R.H. Strotz (1963): "Determinants of Business Investment," in *Impacts of Monetary Policy,* Englewood Cliffs, NJ: Commission on Money and Credit,.pp. 59-233.

Gould, J.P. (1968): "Adjustment Costs in the Theory of Investment of the Firm," *Review of Economic Studies,* 35(1), pp. 47-55.

Lucas, R.E. (1967): "Optimal Investment Policy and the Flexible Accelerator," *International Economic Review,* 8(1), pp. 78-85.

Tobin, J. (1969): "A General Equilibrium Approach to Monetary Theory," *Journal of Money, Credit and Banking,* 1(1), pp. 15-29.

Treadway, A. (1969): "On Rational Entrepreneurial Behavior and the Demand for Investment," *Review of Economic Studies,* 3(2), pp. 227-39.

Wilcoxen, P.J. (1985a): "Numerical Methods for Investment Models with Foresight," IMPACT Preliminary Working Paper No. IP-23, University of

Melbourne.

Wilcoxen, P.J. (1985b): "Computable Models of Investment with Foresight," University of Melbourne, Department of Economics, Working Paper No. 138.

Appendix 1: The Trial Data Set

The data set used to test the model consisted of an eleven-fold replication of the values shown in Table A1.1.

Table A1.1: Variables in the Trial Data Set

Symbol	Definition	Value
K_A	Capital stock A, specific to industry A	1.
ρ	Rental price of nonspecific capital	25
K_B	Capital stock B, nonspecific	10.
K_B^1	Type B capital used by industry 1	3.177778
K_B^2	Type B capital used by industry 2	4.622222
K_B^K	Type B capital used by industry K	2.2
W	Wage rate	1.
L	Total labor supply	5.
L_A^P	Labor used in production by industry A	.25
L_A^I	Labor used in investment by industry A	.042593
L_B	Labor used in investment by industry B	.425926
L_1	Labor used by industry 1	.264815
L_2	Labor used by industry 2	3.466667
L_K	Labor used by industry K	.55
P_K	Price of raw capital goods	1.
P_A	Price of good A	1.
P_1	Price of good 1	1.
P_2	Price of good 2	1.
X_K	Production of raw capital goods	1.1
X_A	Production of good A	.50
X_1	Production of good 1	1.059259
X_2	Production of good 2	4.622222
I_A	Investment by industry A	.1
I_B	Investment by industry B	1.
D_A	Dividends paid by industry A	.121667
D_B	Dividends paid by industry B	1.216667
C	Private consumption	5.404500
G	Government spending	.776981

Table A1.1, Continued: Variables in the Trial Data Set

Symbol	Definition	Value
T_W	Tax on wages	.2
T_S^A	Sales tax on good A	0.
T_S^1	Sales tax on good 1	0.
T_S^2	Sales tax on good 2	0.
T_S^K	Sales tax on good K	0.
TR	Transfer payments	.2
T^D	Dividend tax	.10
T^S	Investment subsidy	.10
γ_1	Technical change parameter, industry 1	.620403
γ_2	Technical change parameter, industry 2	1.240806
γ_K	Technical change parameter, industry K	1.
ρ^e	Exogenous expectations, rental price	.25
W^e	Exogenous expectations, wage rate	1.
P_K^e	Exogenous expectations, raw capital price	1.
P_A^e	Exogenous expectations, price of good A	1.
T^{D*}	Exogenous expectations, dividend tax rate	.10
T^{S*}	Exogenous expectations, investment subsidy	.10
r	Interest rate	.05
ζ	Price deflator	1.

Table A1.2: Parameters in the Trial Data Set

Symbol	Definition	Value
δ	Depreciation rate	.10
θ_A	Investment parameter, industry A	4.259259
θ_B	Investment parameter, industry B	.425926
ϵ_A	Labor exponent, industry A	.5
ϵ_1	Labor exponent, industry 1	.25
ϵ_2	Labor exponent, industry 2	.75
ϵ_K	Labor exponent, industry K	.5
α_C^A	Share of private consumption, good A	.080887
α_C^1	Share of private consumption, good 1	.171360
α_C^2	Share of private consumption, good 2	.747753
α_G^A	Share of government spending, good A	.080887
α_G^1	Share of government spending, good 1	.171360
α_G^2	Share of government spending, good 2	.747753

Appendix 2: The Model's Equations

A2.1 The Investment Submodel

Short run profit on a unit of K_A

$$\beta = \left(\frac{1-\epsilon_A}{\epsilon_A}\right)\left(\frac{\epsilon_A P_A}{W}\right)^{\frac{1}{1-\epsilon_A}}$$

Description of the intertemporal path of K_A

$$\ddot{K}_A - \dot{K}_A\left(r-g-\frac{\dot{W}}{W}\right) - \delta K_A\left(r+\delta-g-\frac{\dot{W}}{W}\right) =$$

$$\frac{P_K}{2W\theta_A}\left((r+\delta-g) - \frac{\dot{P}_K}{P_K} - \frac{\beta}{P_K(1-T^S)}\right)$$

Growth rate of the dividend cost of investment

$$g = -\frac{\dot{T}^S}{(1-T^S)} - \frac{\dot{T}^D}{(1-T^D)}$$

Description of the intertemporal path of K_B

$$\ddot{K}_B - \dot{K}_B\left(r-g-\frac{\dot{W}}{W}\right) - \delta K_B\left(r+\delta-g-\frac{\dot{W}}{W}\right) =$$

$$\frac{P_K}{2W\theta_B}\left((r+\delta-g) - \frac{\dot{P}_K}{P_K} - \frac{\rho}{P_K(1-T^S)}\right)$$

A2.2 The General Equilibrium Submodel

Labor demanded for production in sector A

$$L_A^P = \left(\frac{\epsilon_A P_A}{W}\right)^{\frac{1}{1-\epsilon_A}} K_A$$

Output of sector A

$$X_A = (L_A^P)^{\epsilon_A}(K_A)^{1-\epsilon_A}$$

Labor demanded for investment by sector A

$$L_A^I = \theta_A I_A^2$$

Pretax dividends of sector A

$$D_A = P_A X_A - WL_A^P - \left(P_K I_A + WL_A^I\right)(1-T^S)$$

Labor demanded by sector B

$$L_B = \theta_B I_B^2$$

Pretax dividends of sector B

$$D_B = \rho K_B - (P_K I_B + WL_B)(1-T^S)$$

Labor demanded by sector $i \in \{1,2,K\}$

$$L_i = \frac{1}{\gamma_i} X_i \left(\frac{\rho\epsilon_i}{W(1-\epsilon_i)}\right)^{1-\epsilon_i}$$

Capital B demanded by sector $i \in \{1,2,K\}$

$$K_B^i = \frac{1}{\gamma_i} X_i \left(\frac{W(1-\epsilon_i)}{\rho\epsilon_i} \right)^{\epsilon_i}$$

Zero-profit condition for sector $i \in \{1,2\}$

$$X_i P_i = W L_i + \rho K_B^i$$

Zero-profit condition for sector K

$$X_K P_K = T_S^K (W L_K + \rho K_B^K)$$

Labor market equilibrium condition

$$L = L_A^P + L_A^I + L_B + L_1 + L_2 + L_K$$

Capital B market equilibrium condition

$$K_B = K_B^1 + K_B^2 + K_B^K$$

Consumption

$$C = WL(1-T^W) + (D_A + D_B)(1-T^D) + TR$$

Government spending

$$\begin{aligned} G = {} & T^D(D_A+D_B) - T^S(P_K(I_A+I_B) + W(\theta_A I_A^2 + \theta_B I_B^2)) \\ & + T_S^A P_A X_A + T_S^1 P_1 X_1 + T_S^2 X_2 P_2 + T_S^K P_K X_K \\ & + T^W WL - TR \end{aligned}$$

Market equilibrium for good $i \in \{A, 1, 2\}$

$$X_i = \frac{\alpha_C^i C + \alpha_G^i G}{P_i T_S^i}$$

Market equilibrium for good K

$$X_K = I_A + I_B$$

Investment by sector A

$$I_A = \dot{K}_A + \delta K_A$$

Investment by sector B

$$I_B = \dot{K}_B + \delta K_B$$

Price deflator

$$\zeta = \frac{X_A P_A T_S^A + X_1 P_1 T_S^1 + X_2 P_2 T_S^2}{X_A (P_A T_S^A)_0 + X_1 (P_1 T_S^1)_0 + X_2 (P_2 T_S^2)_0}$$

A2.3 Expectations Formation

Wage

$$W^e = (W)^{\lambda_N}(W^z)^{1-\lambda_N}$$

Rental price of capital B

$$\rho^e = (\rho)^{\lambda_N}(\rho^z)^{1-\lambda_N}$$

Price of output A

$$P_A^e = (P_A)^{\lambda_N}(P_A^z)^{1-\lambda_N}$$

Price of raw capital

$$P_K^e = (P_K)^{\lambda_N}(P_K^z)^{1-\lambda_N}$$

Dividend tax

$$TD^e = (TD)^{\lambda_X}(TD^z)^{1-\lambda_X}$$

Investment subsidy

$$TS^e = (TS)^{\lambda_X}(TS^z)^{1-\lambda_X}$$