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COMPUTABLE MODELS OF INVESTMENT WITH FORESIGHT

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1. INTRODUCTION

Many economic activities explicitly require agents to form and act upon expectations of future conditions. The process of using current information to predict future variables, referred to here as foresight, is often ignored in quantitative models, particularly those which are most concerned with the short run. Even in the short run, however, some variables, such as investment, may be highly elastic with respect to expectations about the future. This poses no problem as long as the range of experiments considered is restricted to those in which agents' expectations could reasonably be assumed to be fixed. In this paper, however, it will be argued that investment, in particular, does respond strongly to changes in expectations, and the additional effort required to model foresight is well justified by the consequent increase in accuracy and applicability of the model.

Improving models in which foresight plays a significant role generally requires solving an intertemporal maximization problem. Setting up the model and finding its first-order conditions is usually feasible; in a qualitative model this is all that is necessary. A quantitative model, however, will need the explicit form of the optimal path, and to find this it will be necessary to solve the system of differential equations given by the first-order conditions. This second step is really the crux of the problem, as it is often impossible to do analytically. For this reason, many models of intertemporal optimization have simply been left as first-order conditions, and only their steady state properties have been studied. The work described here is an effort to proceed further, at least in the study of corporate investment, by using numerical techniques to find the actual optimal path of investment given agents' expectations about future prices and taxes. Although the focus here is on investment, the techniques used are applicable to many other problems involving foresight.

Modelling the effect of foresight on investment provides two major benefits: it increases the domain of possible policy experiments which can be studied with a given model, and it also provides a rigorous treatment of the value of the firm. Expectations will be crucial in determining the difference between temporary and permanent policies; surprise and anticipated policies; and policies carried out over a long period of time in a number of steps. Also, since the market value of the firm is accounted for correctly, it is possible to contrast the windfall gains or losses to equity owners as the result of different policies.

Several small models were developed to assess both the importance of foresight-related effects and the difficulties involved in numerical analysis of such problems. From the results obtained it is clear that information about future policies can have quite a large effect on current variables such as investment. Furthermore, comparisons of numerical and analytical solutions, where possible, indicate that accurate numerical results can be obtained easily and inexpensively. In this paper, the importance of foresight will be studied; a companion piece, Wilcoxen (1985), hereafter referred to as the "technical paper," discusses computational considerations in detail.

1.1 Notational Conventions

The only unusual notation used here is that the time derivative of a variable is denoted by the prime symbol (′) rather than the usual dot. In addition, almost every variable of any importance is an implicit function of time and this dependence will usually be suppressed; the interest rate, for example, will be referred to as r rather than r(t). This also applies to agents' expectations: they expect variables to follow some particular path, not just to be a given value for the rest of time.

2. FORESIGHT

The specific problem considered here is the firm's choice of its investment path to maximize its market value given expected future prices and taxes. Two related topics can be studied: how the firm behaves for a given expected path of exogenous variables, and how its behaviour changes with changes in the information used to form those expectations. This paper deals primarily with the former question: the focus will be on solving the firm's planning problem for a given set of information. First, however, it is useful to make explicit what is meant by expectations.

The current paper explores only a very simple case of expectations formation: agents assign probability one to a particular path of each exogenous variable, the path chosen depending on the current information set. For example, if the government declares that the dividend tax will rise by 10% in ten years and then remain at that level forever, agents believe the tax will do exactly that. Some preliminary results are also presented for cases in which firms never believe government announcements and only react when policies are actually implemented. Intermediate degrees of confidence in government announcements would also be interesting.

Finally, the present analysis is strictly partial equilibrium and confined to investment by a single firm or industry: agents' expectations are not required to be consistent with the consequences of the firm's behaviour. In fact, the outcome of the firm's planning process has no effect on its expectations of future variables; the expected price of the firm's output, for example, is unaffected by the firm's investment plan.

3. MAXIMIZING EXPECTED VALUE

What actually determines investment by real firms is a matter of conjecture, but one plausible assumption is that a firm chooses investment to maximize its current market value. Market value, however, is determined by equity owners' expectations of the firm's future profitability. In solving its optimization problem the firm selects the level of investment which maximizes its stock price given the expectations of its owners. For the information set prevailing at any particular time, the path of these expectations is given for the entire future, so solving the firm's problem will generate a planned investment program lasting forever. As actual time passes, the information set will change and so will the optimal investment program, but the focus here will be on determining a single optimal path for a given set of expectations. In the following sections the formal structure of the value maximization problem will be developed and several models will be presented.

3.1 Problem Structure

In the short run capital is fixed and the firm chooses variable inputs to maximize short run profits. The function giving profit in terms of product price, factor prices, and capital is called the earnings function. Out of earnings the firm may either pay dividends to holders of equity or it may retain the earnings for investment in capital goods. Models in this paper assume the number of outstanding shares of stock is constant and that the firm does not borrow, so investment is entirely determined by retained earnings.

The general short run problem is thus:

$$\max_{L} \quad p(q)q - wL \ ,$$

$$\text{subject to} \quad q = q(L,K) \ ,$$

where L represents all productive factors other than capital. The solution to this problem is an earnings function of the form $E(K,w,P)$.

To make this more concrete, consider the case where q is a Cobb-Douglas function of labour and capital and the firm takes prices as given. If the production function is:

$$q = L^{\varepsilon}K^{1-\varepsilon} \ ,$$

then the earnings function can be shown to be:

$$E(K,w,p) = \left(\frac{1-\varepsilon}{\varepsilon} \right)\left(\frac{\varepsilon p}{w} \right)^{1/1-\varepsilon} wK \ ,$$

which can be written $E(K,w,p) = \beta(w,p)K$ for convenience. This separability is characteristic of production functions which have constant returns to scale, although for other formulations the function β will be different. Finally, expected future earnings may be computed by taking the expectation of function E at some time in the future, given the current information set. At this point the short run problem is completely solved and what remains is to determine the firm's optimal investment path.

It is assumed here that the firm chooses investment to maximize its market value and, furthermore, that equity owners are indifferent between receiving their return as dividends or capital gains. Although unrealistic for an individual investor, this is not unreasonable for the market as a whole because of the large number of financial instruments which exist to convert capital gains into dividend streams. In practice dividends also have a signalling function because the firm typically has better information about its prospects than its shareholders; this role has been ignored in the current paper, but the models developed could be extended to allow dividend policy to be one of the firm's control variables. Finally, it is not clear that management itself is indifferent between dividends and retained earnings, and this may tend to distort the firm's behaviour in favour of investment. In response, equity owners may demand a relatively constant dividend rate, at the expense of potential capital gains, in order to reduce the moral hazard problem of hired management. For all these reasons, the assumption that dividends and capital gains are equivalent forms of return is unrealistic, and many interesting extensions of the models presented here are possible.

Continuing the development of the optimal investment problem, arbitrage with other assets will ensure that the return to holding the firm's equity will just equal the interest rate, so the following condition holds:

$$r(t)V(t) = D(t) + V'(t) .$$

Solving this differential equation gives the value of the firm as a function of its dividend stream and the interest rate. The equation can be solved using the integrating factor:

$$\exp\left(-\int_0^t r(s)ds \right) ,$$

and the fundamental theorem of the calculus to give:

$$\lim_{u\to\infty} \left[V(u)\exp\left(-\int_0^u r(s)ds \right) \right] - V(t)\exp\left(-\int_0^t r(s)ds \right) =$$
$$-\int_t^\infty D(u)\exp\left(-\int_0^u r(s)ds \right)du .$$

The present value of the firm will be required to be finite as time goes to infinity, so the leftmost term must be zero (or constant) and the firm's value must grow more slowly than the interest rate. Technically, this is known as a transversality condition and its implications are discussed in section 4.3. Setting the limit to zero gives the following expression for the value of the firm:

$$V(t) = \int_t^{\infty} D(u)\exp\left(-\int_t^{u} r(s)ds \right)du \ .$$

Recognizing that the dividend stream is an expectation (as is the interest rate path) determined by the information set Ω which prevailed when planning was done, the expression may be written:

$$V(t;\Omega_\tau) = \int_t^{\infty} \xi\left(D(u)\exp\left(-\int_t^{u} r(s)ds \right) \middle| \ \Omega_\tau \right)du \ .$$

Assuming for algebraic simplicity that r is expected to be constant, the value function may be written compactly as:

$$V_\tau(t) = \int_t^{\infty} D_\tau(u)\exp(-r_\tau(u-t))du \ ,$$

where subscripts indicate variables which are expectations conditioned by information set Ω.

Dividends are the residual remaining after the firm allocates its expenditure on investment, C(I,K):

$$D(t) = E(\ K(t),w(t),p(t)\ ) - C(\ I(t),K(t)\ ) \ .$$

The investment cost function is of fundamental importance because its form determines the firm's behaviour. In the simplest case, with no costs to changing the capital stock other than buying the capital goods, C(I,K) is just:

$$C(I) = P_k I \ .$$

More interesting is the behaviour arising from cost functions which include a premium for changing the capital stock; a typical example is:

$$C(I) = P_k I(\ 1 + \phi(I)\ )\ ,$$

where ϕ is usually taken to be increasing in investment. Here the firm is penalized for high levels of investment, and this additional cost may slow the firm's attainment of a given level of capital. Both the form of the earnings function and that of the investment cost function will be examined in more detail but first it is useful to solve a simple model to make the discussion more concrete.

3.2 Example: Zero Adjustment Costs

The firm's long run problem is to select its investment path so as to maximize its expected value at all points in time. Formally, it must solve:

$$\max_{I()} \int_t^{\infty} D_\tau(u)\exp(\ -\int_t^{u} r_\tau(s)ds\ )du\ ,$$
$$\text{subject to} \quad K' = I - \delta K\ .$$

(An additional constraint often made is that investment be non-negative; here it will be assumed that such a constraint would be non-binding, which will be valid as long as the investment path generated is actually positive.) If the interest rate is expected to be constant in the future the maximand is just:

$$\int_t^{\infty} D_\tau(u)\exp(\ -r_\tau(u-t)\ )du\ .$$

Inserting the profit and cost functions from above gives the following problem:

$$\max_{I()} \int_t^{\infty} (\ \beta_\tau(w,p)K - P_k I\ )\exp(\ -r_\tau(u-t)\ )du\ ,$$
$$\text{subject to} \quad K' = I - \delta K\ .$$

This is a straightforward optimal control problem which may be solved by forming the Hamiltonian and finding the first-order conditions as shown:

$$H = (\ \beta_\tau(w,p)K - P_k I\ )\exp(\ -r_\tau(u-t)\ ) + \Lambda(\ I - \delta K\ )\ ,$$

$$\frac{\partial H}{\partial I} = -P_k\exp(\ -r_\tau(u-t)\ ) + \Lambda = 0\ ,$$

$$\frac{\partial H}{\delta K} = \beta_\tau\exp(\ -r_\tau(u-t)\ ) - \delta\Lambda = -\Lambda'\ ,$$

$$\frac{\partial H}{\delta \Lambda} = I - \delta K = K'\ .$$

The multiplier Λ is a function of time which gives the effect on the firm's present value of a change in its capital stock at some time in the future. The problem can be simplified by introducing another function, λ, which gives the effect of an extra unit of capital on the current value of the firm. Thus:

$$\Lambda = \lambda\exp(\ -r_\tau(u-t)\ )\ ,$$

and the first-order conditions become:

$$\lambda = P_k\ , \tag{3.2.1}$$

$$\lambda' = (\ r_\tau+\delta\ )\lambda - \beta_\tau\ , \tag{3.2.2}$$

$$K' = I + \delta K\ . \tag{3.2.3}$$

Because of their simple structure, these equations can be solved analytically. The solution to the second equation is obtained by separating variables, multiplying by the appropriate integrating factor, and integrating to obtain:

$$\lambda(t) = \int_t^\infty \beta_\tau\exp(\ -(r_\tau+\delta)(u-t)\ )du\ .$$

The multiplier λ is the present value of the stream of marginal earnings of an extra unit of capital, divided by the user cost. To see this, evaluate the integral when β is constant, assuming that the multiplier grows more slowly than $r+\delta$, to obtain the following:

$$\lambda(t) = \frac{\beta_\tau}{(r_\tau+\delta)}\ . \tag{3.2.4}$$

Recall that β is the expected marginal value of an extra unit of capital, and that $r+\delta$ is the user cost of one unit of capital. In a price-taking equilibrium, the firm will buy capital until either the price of capital or the price of the firm's output changes. Then equation 3.2.1 will be satisfied: λ will just equal the cost of an additional unit of capital.

3.3 The General Problem

Having studied the simple examples above, it is useful to develop a general structure which may be applied to analyze a number of specific models. Attention will be focused on the class of models for which the firm's investment problem is of the form:

$$\max_{I()} \int_t^{\infty} \left(E_\tau(K) - C_\tau(I,K) \right)(1-T_\tau^d)\exp\left(-\int_t^u r_\tau ds\right)du, \qquad (3.3.1)$$

$$\text{subject to} \quad K' = I - \delta K, \qquad (3.3.2)$$

where T^d is the tax on dividends. Resolving the expectation operation can be fairly difficult depending on the structure of agents' expectations. The previous example avoided this problem because the agents had very simple expectations about the future; they believed certain parameters would hold with probability one. An interesting extension would be to consider expectations which have a non-trivial distribution: the government's policy announcements are assigned a probability less than one of being enacted. This would be particularly relevant in the case of a policy for which the implementation date is far in the future. Implementing the policy gradually using a number of small steps is one way to establish credibility, and a numerical example of this will be discussed in section 5.3.3. In the rest of the paper agents will expect a particular path of exogenous variables with probability one.

In the above problem no formal restriction is made on the sign of investment, so the models considered are applicable only when gross investment is positive and capital leaves the industry only by depreciation. Relaxing this restriction would be interesting because some sort of market for used capital often exists; disinvestment occurs when the dividend stream from the marginal unit of capital falls below its sales price. As in the model above, it will be assumed that gross investment will be positive and a formal constraint, had one been specified, would be satisfied automatically.

Forming the Hamiltonian and finding the first-order conditions in current value form gives the following:

$$\lambda = \left(\frac{\partial C_\tau}{\partial I} \right)(1-T_\tau^d), \tag{3.3.3}$$

$$\lambda' = (r_\tau+\delta)\lambda - \left(\frac{\partial E_\tau}{\partial K} - \frac{\partial C_\tau}{\partial K} \right)(1-T_\tau^d), \tag{3.3.4}$$

$$K' = I - \delta K. \tag{3.3.5}$$

These conditions apply to any problem of the form given by equations 3.3.1 and 3.3.2 and, when combined with a number of boundary conditions, provide a full, albeit implicit, description of the optimal path of investment over time. The next step is to find the explicit form of the investment path by inserting appropriate earnings and investment cost functions into the conditions above, and solving the resulting system of differential equations. In general, this turns out to be substantially more difficult than finding the first-order conditions; in essence the process is integration rather than differentiation.

The first model developed is only slightly more complicated than the zero-adjustment cost model above, but it produces striking results. The earnings function again has constant returns to scale, but the investment cost function will be modified to include linear adjustment costs.

3.4 Constant Returns to Scale Model

Much more interesting than the zero adjustment cost case is the situation which arises when investment costs are not linear in investment. In general, this will cause it to be suboptimal for the firm to jump immediately to the new equilibrium in the event of a change in expectations. The model constructed in this section is very simple yet it allows a number of interesting experiments to be performed regarding changes in expectations, and illustrates the effect of nonlinear investment costs.

To begin with, suppose the investment cost function is given by:

$$C(I) = P_k I(\ 1 + \phi(I)\ )(1-T^s_\tau),$$

where T^s is an investment subsidy provided by the government to lower the purchase price of capital. Now consider the case where adjustment costs are linear in investment, so ϕ may be written as shown:

$$\phi(I) = \gamma + \theta I.$$

Inserting this into the cost function gives the following:

$$C(I) = P_k I(\ 1+\gamma\ ) + \theta P_k I^2.$$

The earnings function used will be the same as that of section 3.2:

$$E(K) = \beta_\tau K.$$

The general first-order conditions derived in the previous section may now be applied to generate this model's first-order conditions, which are shown below:

$$\lambda = (\ P_k(1+\gamma) + 2\theta P_k I\ )(1-T^s_\tau)(1-T^d_\tau), \qquad (3.4.1)$$

$$\lambda' = (r_\tau+\delta)\lambda - \beta_\tau(1-T^d_\tau), \qquad (3.4.2)$$

$$K' = I - \delta K. \qquad (3.4.3)$$

The second equation may be solved directly to obtain the function λ over time; the first equation then is used to find investment as a function of λ. In the previous model, the firm would simply jump to the new optimal level of capital if a change in β and hence λ occurred. Now, new investment will be finite and determined by λ. Equation 3.4.1 can be solved explicitly for a function $I(\lambda)$:

$$I = \frac{1}{2\theta} \left(\frac{\lambda}{P_k(1-T_\tau^s)(1-T_\tau^d)} - (1+\gamma) \right). \tag{3.4.4}$$

The term λ/P_k has special significance because it is the ratio of the firm's market value to its capital cost at the margin. Suppose for the moment that β is expected to be constant in the future. Then, following the technique used in section 3.2, the condition below may be derived:

$$\frac{\lambda}{P_k} = \frac{\beta_\tau(1-T_\tau^d)}{P_k(r_\tau+\delta)},$$

where $P_k(r\tau+\delta)$ is the cost of one unit of capital measured in dollars. Thus λ/P_k is the change in the firm's value for a one dollar change in its book value, the ratio called marginal (current value) q.

Two important observations should be made regarding the function q. First, the model does not have the usual property that q is equal to one in the steady state. This is because of both the government subsidy and the constant term in the adjustment cost function. If there were no subsidy and no constant term, it would be possible for the firm to reduce its adjustment costs to zero by investing very slowly. In this case the marginal value of an extra unit of capital would, at equilibrium, have to equal the price of an additional capital good, so marginal q would be one. If this property were desired, all that would be necessary is to set the subsidy and the value of γ to zero.

Secondly, the model is not homogeneous in investment and capital. This deficiency may be unimportant in a microeconomic analysis of a single firm, but it would be of consequence when modelling the economy as a whole, since it means that adjustment costs will not be independent of the scale of the economy. The difficulties generated by modifying the investment cost function to have this homogeneity property will be discussed in section 6.1, and a model which incorporates the feature will be presented. A number of interesting results can be obtained from simpler models, and the primary focus will be on them.

Returning to the development of current model, one interesting property it possesses as a result of the CRTS production function is that q is constant for a given expectation β_τ; thus investment is also constant. In equilibrium, the amount of investment will be just equal to the depreciation rate times the steady state capital stock. Using this property, the steady state is obviously:

$$K^{ss} = \frac{1}{2\delta\theta} \left(\frac{\beta_\tau}{P_k(r_\tau+\delta)(1-T^s_\tau)} - (1+\gamma) \right). \tag{3.4.5}$$

When good news arrives, β changes, altering the equilibrium steady state capital stock. Investment changes immediately to the depreciation rate times the steady state, and capital approaches the steady state asymptotically. The explicit solution can be found by integrating the constraint with constant investment:

$$K' = \delta(K^{ss} - K),$$

which results in the following optimal path of capital as a function of time:

$$K(t) = K^{ss} - (K^{ss} - Ko)\exp(-\delta t), \tag{3.4.6}$$

where Ko is the original capital stock. The difference between the current stock and the steady state value clearly falls at rate δ. This model would apply, for example, to the case of a surprise sales tax which lowers the firm's received price. At implementation, investment drops and the firm's capital stock decays toward the new equilibrium level.

Another interesting feature of this model is that the firm is unable to exploit advance knowledge of an increase in the investment subsidy. This arises because the second first-order equation alone determines λ, and it is independent of the subsidy. Consequently, λ is determined by the interest rate, the marginal product of capital, and the dividend tax; investment depends on λ and on the subsidy. This implies that if the government announces an increase in the subsidy, the firm will be unable to take advantage of the information and will behave exactly as though surprised at the time of implementation. The subsidy does not change λ before implementation, so the firm does not postpone any investment. Once the subsidy takes effect, investment will change because λ has remained constant but $(1-T^s)$ has decreased. While the marginal cost of post-subsidy capital will be lower at the old optimal level of investment, the marginal cost of an extra unit at the new optimal level is at least λ. The model does respond to other announced policies, in particular to changes in the dividend tax as discussed below.

3.4.1 The Dividend Tax Experiment

One important policy which produces a marked effect on the constant returns model is a change in the dividend tax rate. Forecasts of higher dividend taxes will influence the time path of λ and thus investment. Suppose at time zero it is announced that the dividend tax will be raised at time T. Recall that λ is given by:

$$\lambda(t) = \int_t^\infty \beta_\tau(1-T_\tau^d)\exp(\ -(r_\tau+\delta)(u-t)\ )du.$$

Integrating separately over the periods before and after implementation of the policy gives:

$$\lambda^1(t) = \int_t^T \beta_\tau(1-T_1^d)\exp(\ -(r_\tau+\delta)(u-t)\ )du\ +$$

$$\int_T^\infty \beta_\tau(1-T_2^d)\exp(\ -(r_\tau+\delta)(u-t)\ )du,$$

when t<T; for t>T the integral is just:

$$\lambda^2(t) = \int_t^\infty \beta_\tau(1-T_2^d)\exp(\ -(r_\tau+\delta)(u-t)\ )du.$$

Because the two taxes are constant over their respective intervals, evaluation is straightforward and results in the functions:

$$\lambda^1(t) = \frac{\beta_\tau(1-T_1^d)}{(r_\tau+\delta)}\left(\ 1 + \left(\frac{T_1^d-T_2^d}{1-T_1^d}\right)\exp(\ -(r_\tau+\delta)(T-t)\ )\ \right),$$

and

$$\lambda^2(t) = \frac{\beta_\tau(1-T_2^d)}{(r_\tau+\delta)}.$$

Notice that $\lambda^1(T) = \lambda^2(T)$: there are no capital losses at implementation because the value of the firm already reflects the new tax. However, λ does jump at the time of announcement from:

$$\lambda^0(0) = \frac{\beta_\tau(1-T_1^d)}{(r_\tau+\delta)},$$

to

$$\lambda^1(0) = \lambda^0(0)\left(\ 1 + \frac{(T_1^d-T_2^d)\exp(\ -(r_\tau+\delta)T\ )}{(1-T_1^d)}\ \right).$$

Because the tax is increasing, the market value of the firm drops at announcement.

The consequences for investment can be found by using the investment function shown in equation 3.4.4. Before the new policy is announced, investment is given by:

$$I^0(t) = \frac{1}{2\theta}\left(\frac{\beta_\tau}{P_k(1-T_1^s)(r_\tau+\delta)} - (1+\gamma) \right). \qquad (3.4.1.1)$$

After the announcement but before implementation, investment is:

$$I^1(t) = \frac{1}{2\theta}\left(\frac{\beta_\tau}{P_k(1-T_1^s)(r_\tau+\delta)}\left(1+\left(\frac{T_1^d-T_2^d}{1-T_1^d}\right)\exp(-(r_\tau+\delta)(T-t))\right)-(1+\gamma)\right), \qquad (3.4.1.2)$$

while in the post-implementation period it is just:

$$I^2(t) = \frac{1}{2\theta}\left(\frac{\beta_\tau}{P_k(1-T_2^s)(r_\tau+\delta)} - (1+\gamma) \right). \qquad (3.4.1.3)$$

At implementation investment jumps back to its initial steady state level because it is independent of the dividend tax as long as the tax is expected to be constant. A jump also occurs at announcement and can be found by comparing equations 3.4.1.1 and 3.4.1.2 at time zero. Finally, the path of the capital stock itself may be calculated by integrating investment. Given an initial capital stock Ko, the stock at time t is given by:

$$K(t) = Ko\cdot\exp(-\delta t) + \int_0^t I(s)\exp(\delta(s-t))ds.$$

Substituting in investment in the pre-implementation period and integrating yields the optimal path of the capital stock over the period between announcement and implementation of the policy. Subtracting from that the amount of capital there would have been if the dividend tax had not changed gives the following expression, valid for t<T:

$$\Delta K^1(t) = \frac{\beta_\tau(T_1^d-T_2^d)(\exp(-(r_\tau+2\delta)(T-t))-\exp(-(r_\tau+2\delta)T))}{2\theta P_k(r_\tau+\delta)(1-T_1^s)(1-T_1^d)(r_\tau+2\delta)}. \qquad (3.4.1.4)$$

The overall sign is given by the difference in the taxes because the denominator will be positive as long as investment is positive. Interestingly, the magnitude of the drop at any particular time is increasing (asymptotically) in T. This occurs because with longer advance warning it is optimal to begin drawing larger dividends from the time of announcement. Also, the drop is independent of the initial capital stock, provided the constraint that investment be non-negative is satisfied at all times.

Calculation of the firm's value may be accomplished by integrating the dividend stream corresponding to the above paths of capital and investment. The algebra is tedious, so the general case result is omitted. A numerical example will be computed in a later section.

In words the response of the model is as follows. The shadow value of capital, λ, will decrease up to implementation provoking a decline in investment and a fall in the capital stock. Equity owners compensate for the tax by taking high dividends before implementation. After implementation investment jumps back to its equilibrium value and the capital stock asymptotically approaches the steady state. This is the familiar result that a pure profits tax does not change the optimal mix of inputs. The announcement effect arises purely from the dynamic evolution of the model - there is no change in the steady state.

The dividend tax experiment is useful because it is simple, analytically tractable and provides a striking example of the importance of foresight in the determination of investment. Another problem which can be solved by similar methods is the response of investment to anticipated changes in the interest rate, and this problem is developed in the following section.

3.4.2 Interest Rate Experiments

No particular difficulty is imposed by allowing the expected interest rate to be a function of time rather than constant. An experiment with a single jump in the interest rate can be solved by the techniques used above - separating the future into a "before" and an "after" period. The solution to such an experiment can be found as follows. First, recall equation 3.4.2, one of the model's first order conditions:

$$\lambda' = (r_\tau+\delta)\lambda - \beta_\tau(1-T^d_\tau).$$

Making r a function of time does not link the first order conditions, so this equation may be solved in isolation to obtain the following function:

$$\lambda(t) = \int_t^\infty \beta_\tau(1-T^d_\tau)\exp(- \delta(u-t) - \int_t^u r_\tau(s)ds).$$

If the interest rate changes once from r^1 to r^2 at time T, this equation can be replaced by:

$$\lambda^1(t) = \beta_\tau(1-T^d_\tau)(\int_t^T \exp(-(r_1+\delta)(u-t))du + \int_T^\infty \exp(-(r_2+\delta)(u-t))du),$$

when t<T. Integrating and collecting terms yields:

$$\lambda^1(t) = \frac{\beta_\tau(1-T^d_\tau)}{(r_1+\delta)}(1-\exp(-(r_1+\delta)(T-t))+(\frac{r_1+\delta}{r_2+\delta})\exp(-(r_2+\delta)(T-t))),$$

for region 1 where t<T; in region 2, t>T the following holds:

$$\lambda^2(t) = \frac{\beta_\tau(1-T^d_\tau)}{(r_\tau+\delta)}.$$

In addition, the derivative of λ can be shown to be:

$$\lambda'^1 = \beta (1-T^d_\tau)(\exp(-(r_2-r_1)(T-t)) - 1)\exp(-(r_1+\delta)(T-t)).$$

Clearly $\lambda'(T)$ is zero. For all t<T, the sign of λ' will be the same as the

sign of $-(r^2-r^1)$ so λ is monotonic; investment will be of the same sign as λ, so it is also monotonic. It is useful to calculate the difference between the value of λ prior to implementation and its value if the rate change had been a surprise at time zero. At the time of a surprise λ jumps immediately to its region 2 value so the difference made by the announcement is given by:

$$\Delta\lambda^A(t) = \beta_\tau(1-T_\tau^d)\left(\frac{1-\exp(-(r_1+\delta)(T-t))}{r_1+\delta} - \frac{1-\exp(-(r_2+\delta)(T-t))}{r_2+\delta} \right).$$

Differentiating with respect to time gives the result obtained above for λ' (since λ^2 is a constant), so if $r^2>r^1$ then λ' is negative and so is $(\Delta\lambda)'$. Because λ is weakly monotonic and $\Delta\lambda$ is zero at T, $\Delta\lambda$ must be non-negative before T. Investment will be higher in the pre-implementation period than it would have been if the increase had come as a surprise; λ and thus investment will be the same after the change. The difference between λ with the announced policy and the λ that would have prevailed if no change had occurred is just:

$$\Delta\lambda^B(t) = \beta_\tau(1-T_\tau^d)\left(- \frac{\exp(-(r_1+\delta)(T-t))}{r_1+\delta} + \frac{\exp(-(r_2+\delta)(T-t))}{r_2+\delta} \right).$$

If $r^2>r^1$, $\Delta\lambda$ will be negative, so both firm value and investment will be lower than if no change had occurred. The effect of the announcement is to maintain higher levels of investment in the period before the rate increases (but lower than what it would have been if no increase occurred). The implications for the capital stock, dividends and market value are obvious, and λ could be used as before to find analytical results if desired. Thus, problems in which the interest rate varies provide no conceptual difficulties.

So far, all discussion has been confined to models in which the earnings function has constant returns to scale. As noted above, this produces the peculiar result that anticipation of an investment subsidy has no effect on the optimal path of investment. Introducing diminishing returns to scale will cause this to change, as discussed in the following section.

3.5 Diminishing Returns

In contrast to the previous model, now suppose that the firm faces a downward sloping demand curve rather than taking prices as given. Furthermore, assume investment costs do not depend on capital to isolate the effect. If the earnings function has the form:

$$E(K) = \beta_\tau K - (\alpha_\tau/2)K^2,$$

the general first-order conditions above become:

$$\lambda = \left(\frac{\partial C_\tau}{\partial I} \right)(1-T_\tau^d), \tag{3.5.1}$$

$$\lambda' = (r_\tau+\delta)\lambda - (\beta_\tau - \alpha_\tau K)(1-T_\tau^d), \tag{3.5.2}$$

$$K' = I - \delta K. \tag{3.5.3}$$

This shows clearly that λ depends on K, but K depends on investment and hence λ through condition 3.5.1; previously, λ was independent of K. Solving this model requires solving 3.5.1 and 3.5.2 simultaneously, either as a system of two first-order differential equations or as a single second-order equation. This will be discussed in more detail in section 4.1.

Intuitively, the difference between this model and its predecessors is that in the unlinked model, a change in current investment has no effect on next period's λ and hence no effect on the optimal amount of investment tomorrow. With diminishing returns however, less investment today lowers the capital stock tomorrow and thus raises λ, increasing the optimal amount of investment in the future. Unlike the constant returns case, this model will respond to announced changes in the investment subsidy.

4. BOUNDARY VALUE PROBLEMS

The models constructed so far have all involved solving a system of differential equations (the first-order conditions) subject to certain conditions. Technically, they are examples of boundary value problems and as such they have a number of interesting properties which will be discussed in this section. Fundamentally, the problem is to solve a system of three equations:

$$\lambda = f(I,K,x), \qquad (4.1)$$

$$\lambda' = g(\lambda,K,x), \qquad (4.2)$$

$$K' = h(K,K'',x), \qquad (4.3)$$

(where x is a vector of exogenous variables and expectations) subject to two boundary conditions which arise essentially as constants of integration. The models considered so far have been from a special class for which the above equations have the form:

$$\lambda = \hat{f}(I,x),$$

$$\lambda' = \hat{g}(\lambda,x),$$

$$I = \hat{h}(K,K',x).$$

The second equation was solved for λ, the result inserted into the first equation, from which investment was found. Investment was then inserted into the final equation to find the path of the capital stock. One of the two boundary conditions was just the initial capital stock while the other was the requirement that the value of the multiplier grew more slowly than $r+\delta$. This approach, as noted in the diminishing returns case, will generally not be applicable because in many models the first-order conditions will be linked.

To analyze the problem, it is helpful to reduce the system of equations by using the third equation to eliminate investment from the other two. The result is:

$$\lambda = f(h(K,K',x),K,x),$$

$$\lambda' = g(\lambda,K,x).$$

These may be solved simultaneously in their current form, or they may be combined to yield a single second-order differential equation in capital by eliminating λ. Differentiate the first equation and substitute into the second to get:

$$f'(h(K,K',x),K,x) = g(f(h(K,K',x),K,x),K,x).$$

This general formula is somewhat complicated and provides no insight; fortunately the equation for any particular model is much simpler and quite a bit more intuitive. The appropriate equation for the models discussed above will be derived and analyzed. Not only does the equation provide useful insight into the model's dynamic behaviour, but it is particularly easy to implement numerically.

4.1 Models in Second-Order Form

Although it is mathematically equivalent to express the model as two first-order differential equations or as a single second-order equation, the latter form emphasizes some of the properties of the solution more clearly, and it is worthwhile to discuss it in depth. Unfortunately, this introduces a certain amount of ambiguity because the second-order form of the model is not to be confused with the second-order conditions for optimality. Throughout this paper, the term "second-order form" will refer to the differential equation derived from combining the two first-order conditions; it will be assumed that the solution found is in fact a maximum and the second-order condition satisfied.

To illustrate the difference made by converting a model into second-order form, it is convenient to focus on the CRTS model. Recall the first-order conditions from section 3.4:

$$\lambda = (P_k(1+\gamma) + 2\theta P_k I)(1-T^s_\tau)(1-T^d_\tau), \tag{4.1.1}$$

$$\lambda' = (r_\tau+\delta)\lambda - \beta_\tau(1-T^d_\tau), \tag{4.1.2}$$

$$K' = I - \delta K. \tag{4.1.3}$$

Assuming for simplicity that the price of capital is constant, the first equation may be differentiated with respect to time to give the following:

$$\lambda' = 2\theta I' P_k(1-T^s_\tau)(1-T^d_\tau) + P_k(1+\gamma+2\theta I)(-T^{s'}_\tau(1-T^d_\tau)-T^{d'}_\tau(1-T^s_\tau)). \tag{4.1.4}$$

Similarly, the accumulation constraint may be differentiated to give:

$$K'' = I' - \delta K. \tag{4.1.5}$$

The second-order form is derived by substituting equations 4.1.1 and 4.1.4 into equation 4.1.2 to eliminate λ; investment is then eliminated using equations 4.1.3 and 4.1.5. Collecting terms in K and dividing through by $2\theta P_k(1-T^s)(1-T^d)$ gives the final equation shown below:

$$K'' + K'(g-r_\tau) + \delta K(g-r_\tau-\delta) = \frac{1}{2\theta}\left(-(1+\gamma)(g-r_\tau-\delta) - \frac{\beta_\tau}{P_k(1-T^s_\tau)}\right), \tag{4.1.6}$$

where

$$g = -\frac{T^{s'}_\tau}{1-T^s_\tau} - \frac{T^{d'}_\tau}{1-T^d_\tau}.$$

The interpretation of g is interesting and straightforward. The first term is the rate of growth of the firm's share in investment costs (costs exceed the firm's share because of government subsidy). Similarly, the second term is the growth rate of the post-tax fraction of dividends. The sum of the two is the rate of growth of the dividends cost of a given level of investment. To see how this works suppose the subsidy is increased while the dividends tax is held constant: g will be negative, indicating that the amount of dividends given up for a specific level of investment is decreasing. The same result holds if the tax is increasing: a dollar invested means giving up a smaller and smaller amount of dividends as the tax increases. The function g plays an important role in the solution to the model. Notice that it is also the only way the dividend tax enters the problem. In most simulations, taxes and subsidies will be changed in a small number of discrete jumps and g will be zero everywhere except a finite number of points at which it will be undefined. Somewhat surprisingly this will have tremendous implications for the optimal path of the capital stock.

Second order equation 4.1.6 is the solution to the firm's market value maximization problem, but to find the actual path of the planned capital stock explicitly, it is necessary to solve the equation for capital as a function of time. Generally, this will be difficult to do analytically because the coefficients of the capital terms will be functions of time. It is quite possible to solve numerically, as discussed below, but first it is worthwhile to examine the properties of the equation itself.

Suppose the interest rate is expected to be constant at r and no changes in taxes or subsidies are anticipated. Then the equation becomes:

$$K'' - r_\tau K' - \delta(r_\tau+\delta)K = \frac{1}{2\theta}\left((1+\gamma)(r_\tau+\delta) - \frac{\beta_\tau}{P_k(1-T^s_\tau)} \right). \qquad (4.1.7)$$

As with all second-order differential equations with constant coefficients, finding the solution is accomplished in two steps: first a solution is found to the homogeneous equation:

$$K'' - r\,K' - \delta(r+\delta)K = 0,$$

then a particular solution which satisfies the full equation is determined; the general solution is the sum of the homogeneous and particular solutions. The homogeneous solution may be found by assuming the form of K(t) is $A\cdot\exp(\sigma t)$. Differentiating and substituting into equation 4.1.7 gives the following characteristic equation for the homogeneous problem:

$$\sigma^2 - r_\tau\sigma - \delta(r_\tau+\delta) = 0.$$

This equation can be factored to give roots:

$$\sigma^1 = r_\tau+\delta,$$

$$\sigma^2 = -\delta.$$

The homogeneous solution is the sum of two exponential solutions with the roots determined above; in this case has the form:

$$K_h = C^1\exp((r_\tau+\delta)t) + C^2\exp(-\delta t).$$

If β and T^s are expected to be constant, a particular solution is the steady state, which can be found by setting K'' and K' to zero in equation 4.1.7:

$$K_p = K^{ss} = \frac{1}{2\delta\theta}\left(\frac{\beta_\tau}{P_k(r_\tau+\delta)(1-T_\tau^s)} - (1+\gamma) \right). \tag{4.1.8}$$

Notice that this is identical to equation 3.4.5 above. (To verify that it is a solution, differentiate it and substitute into the second-order equation.) Combining the homogeneous and particular solutions gives the following general solution to the second-order form equation:

$$K(t) = C^1\exp((r_\tau+\delta)t) + C^2\exp(-\delta t) + K^{ss}.$$

The constants are evaluated using two boundary conditions imposed on the solution and these will be determined below. First, however, it is useful to discuss an important property of solution which makes numerical analysis particularly difficult.

4.2 The Saddle Point Property

The fact that the two roots are real, unequal and of opposite sign means the model possesses a unique path from any initial capital stock to the steady state. This can be illustrated by considering the behaviour of the system when the roots take other forms. If both roots are positive and real the solution will become unbounded at large t except in the trivial case where $C^1 = C^2 = 0$; for no other boundary conditions will the steady state be achieved. If the roots are both negative and real the solution approaches the steady state no matter what the values of C^1 and C^2. Finally, if the roots have a complex component the signs of the real parts determine the general character of the solution as above, but the imaginary term adds oscillations the amplitude of which is governed by the sign of the real part of the root. With real roots of opposite sign, the solution will approach the steady state monotonically if $C^1 = 0$, otherwise it will be unbounded. The path described by the model when C^1 is zero is said to possess the saddle point (path) property. That the solution has this character will make solving the problem numerically considerably more difficult because of the way the boundary conditions are usually specified.

4.3 Imposing the Boundary Conditions

If the model is set up as an initial value problem with both boundary conditions given as values which held at the initial time, for example:

$$K(0) = Ko,$$
$$K'(0) = Ko,$$

no particular difficulty is encountered. The conditions are imposed by solving the following equations for C^1 and C^2:

$$Ko = C^1 + C^2 + K^{ss},$$

$$Ko' = \sigma^1 C^1 + \sigma^2 C^2,$$

which yields the constants shown below:

$$C^1 = (- \sigma^2 Ko + Ko' + \sigma^2 K^{ss})/(\sigma^1 - \sigma^2),$$

$$C^2 = (\sigma^1 Ko - Ko' - \sigma^1 K^{ss})/(\sigma^1 - \sigma^2).$$

This highlights the saddle point nature of the model: the solution will be unbounded except in the case where C^1 is zero, which only occurs when $\sigma^2(Ko-K^{ss}) = K'(0)$; hence there is a unique value of $K'(0)$ for which the solution will converge to the steady state. This implies there is a unique level of investment at time zero; for any other value the system will become unboundedly far from the steady state with increasing time. (Also note that if $K'(0) = \sigma^2(Ko-K^{ss})$, $C^2 = Ko-K^{ss}$, which is a particular feature of this model.)

In economic problems it is generally required that the solution converge to the steady state as time grows, but one of the initial conditions, $K'(0)$ in this case, is unknown. In this case the boundary conditions have the form:

$$K(0) = Ko,$$

$$\lim_{t\to\infty} K(t) = K^{ss}.$$

As discussed above, these imply a unique $K'(0)$. Finding this value is frequently the goal of the problem: to determine what happens to investment now, given some proposed policy. Notice that if $C^1 \neq 0$, the rate of growth of firm value approaches $r+\delta$, so to be consistent with the derivation of firm value in section 3.1, C^1 must be zero. (The condition imposed on the firm's value was actually stronger because it rules out purely speculative growth in addition to unbounded growth of the dividend stream.)

The interpretation of the transversality condition is as follows. If the capital stock will eventually converge to the steady state, $K'(0)$ as found above is the unique value which maximizes the firm's market value; any other choice of $K'(0)$ is suboptimal. Choosing $K'(0)$ too small will require excessive investment later resulting in a smaller present-value dividend stream than if it had been chosen optimally; similar analysis holds for $K'(0)$ too high. The dynamics of the system should not be interpreted to mean that the capital stock will actually explode if $K'(0)$ is not chosen correctly; a better interpretation is if $K'(0)$ is not optimal the firm will have to compensate later at greater cost. The transversality condition simply means there is a unique value-maximizing choice of $K'(0)$ to be made when the tax regime changes.

Imposing the boundary conditions on the simple CRTS model gives the following equation for the capital stock:

$$K(t) = K^{ss} - (K^{ss} - Ko)\exp(-\delta t),$$

where K^{ss} is defined in equation 4.1.8. This is exactly the result obtained in section 3.4 and describes the optimal path of the capital stock when no changes are expected in the model's parameters; it would apply, for example, after the last in a series of tax changes as the capital stock moves toward the steady state. Investment can be easily calculated from the accumulation constraint and turns out to be independent of time:

$$I = \delta K^{ss}.$$

In this model it is optimal, in the absence of expected changes in taxes, to invest at the steady state depreciation level. This will cause the capital stock to move toward the steady state asymptotically if it starts somewhere else. The firm's value can be found by integrating the net dividend stream which gives:

$$V(t) = \left(\frac{\beta_\tau (K(t)-K^{ss})}{(r_\tau+\delta)}\right) + \left(\frac{\beta_\tau K^{ss} - P_k(1+\gamma)\delta K^{ss} - P_k\theta\delta^2 K^{ss^2}}{r_\tau}\right).$$

The term at the far right is the value of the firm at the steady state while the other term is the present value of the dividend stream gained or lost as the capital stock approaches the steady state. Notice that the derivative of value with respect to the current capital stock is just $\beta\tau/(r\tau+\delta)$ which is exactly marginal q as discussed in section 3.2 (measured per unit of capital rather than per dollar of book value).

4.4 The First-Order Form

To solve the model as a system of first-order equations requires the same basic steps outlined above. Boundary conditions for the two equation system will be identical to those for the single equation, but the missing initial value is best thought of as the initial value of λ. The characteristic roots of the two equation system may be found as follows. First, substitute out investment using equation 4.1.3 to get:

$$\lambda = 2\theta P_k(K'+\delta K)(1-T_\tau^s)(1-T_\tau^d),$$

and

$$\lambda' = (r_\tau+\delta)\lambda - \beta_\tau(1-T_\tau^d).$$

Now rearrange the first equation to get:

$$K' = -\delta K + \left(\frac{1}{2\theta P_k(1-T_\tau^s)(1-T_\tau^d)}\right)\lambda.$$

The homogeneous form of the system is found by dropping the constant term from the λ' equation and solving it is accomplished by introducing the trial functions $\lambda = A\cdot\exp(\sigma t)$ and $K = B\cdot\exp(\sigma t)$ into the above equations. Eliminating the exponential, collecting terms and writing the problem in matrix form yields:

$$\begin{bmatrix} -\delta-\sigma & \dfrac{1}{2\theta P_k(1-T_\tau^s)(1-T_\tau^d)} \\ 0 & (r_\tau+\delta)-\sigma \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

This must hold for arbitrary values of the constants, since they will be specified in accordance with the boundary conditions, so the left-most array must be singular and its determinant, zero. This property can be used to find a characteristic equation identical to that obtained from the second-order form.

Since the two forms are mathematically equivalent, the best approach to use depends on the structure of the problem; it may be worth some extra algebra to see the single equation model. For example, notice that the function g discussed above does not appear in the first-order form but does appear in the second-order problem. This function in particular provides a great deal of insight into the role of taxes in the model, insight not provided by the first-order form. For more details regarding the first-order technique refer to Kamien and Schwartz (1981).

4.5 Numerical Methods

Regardless of whether the problem is set up as a single second-order equation or a set of first-order equations, the fundamental nature of the firm's value maximization is such that the first-order conditions form a set of differential equations which must be solved to find the path of investment. For complex problems it will often be impossible to solve for the optimal path analytically, so numerical analysis will be required. If a number of tax changes are expected it is quite difficult to find the solution to even the simple CRTS model discussed above. The problem is not in finding the first-order conditions of the solution, it is in integrating those equations along the path described by the expectations. Even a simple experiment will generate fairly complex expectations and quite complicated integrals of the first-order conditions.

Furthermore, a number of boundary conditions will be required, and some will be only known at the terminal point. This serves to complicate numerical analysis because it precludes simple integration of the equations forward from the initial conditions. For this reason, such problems are known as two-point boundary value problems. A number of methods have been developed to solve them, and these will be discussed here briefly, and in more detail in the technical paper. The existence of a solution to the boundary value problem is assumed here and the focus will be on finding it. It is possible, however, that no solution may exist for some problems and boundary conditions.

Three methods of particular interest will be considered here: Shooting, multiple shooting, and finite differences. The first, simple shooting, is an iterative process in which guessed values of the missing initial conditions are supplied and the equations integrated forward to the terminal time. In general, the terminal value found will differ from the specified boundary value and this miss distance can be used in a generalized Newton-Raphson procedure to update the guessed initial condition. Unfortunately, this technique is particularly vulnerable to the numerical instabilities associated saddle point problems: if the initial guess is only a small distance away from the true value, the exponential growth term will cause the terminal value to be very far from the boundary value. Typically, shooting will fail in saddle point applications.

Multiple shooting is a refinement of the simple shooting technique in which the total time interval is divided into a small number of subintervals with boundary conditions specified at the end of each. The system is then "shot" across the series of intervals which produces a sequence of miss distances at the end of each interval. This is very useful for controlling the explosive nature of the problem, but results in a much larger set of equations which must be solved to find the solution (of the order of the initial number of equations times the number of shooting intervals) and must be applied carefully to avoid instabilities.

The finite difference method avoids all the difficulties of instability associated with shooting or multiple shooting, but at the cost of requiring the solution to an even larger set of equations. All differentials in the original problem are replaced by finite difference approximations, for example:

$$y'(t) \cong \frac{y(t+\Delta t) - y(t-\Delta t)}{2\Delta t} .$$

Accuracy of the solution is determined by the density of grid points and thus the magnitude of Δt. The transformation results in a set of equations for the variable at a discrete set of points. If the original system was linear, the transformed system will be linear and can be solved by any of the usual methods for solving systems of linear equations; if the system was nonlinear it can be solved using Newton's method or some other technique. The strengths of the finite difference method are: (1) it is numerically stable, (2) the approximation error is dispersed throughout the grid rather than concentrated at the terminal point, and (3) that it is easy to solve, particularly if the system is linear. Linearity here refers to the functions being found and not to the solution path over time (no terms of the form K^2 or λK appear in the differential equations). Further discussion and comparison of these techniques is the subject of the technical paper.

The results presented below were obtained using a finite difference algorithm because all of the models considered were linear, and the extra work required to put the model in finite difference form was more than offset by the consequent gain in computational speed and reduction in cost. Furthermore, this approach is well suited to integration into Johansen-style general equilibrium models, as it employs the same solution procedure. Also, all models were solved in second-order form and discussion of them will be in that context.

One further aspect of numerical solution is the technique used to impose the terminal time boundary condition. Obviously it is impossible to model time approaching infinity, so an approximation to the steady state condition must be made. The method used was to require that the model reach the true steady state at some finite but large time, say 100 years in the future. Further discussion of this approximation is contained in the technical paper, but in most cases the error introduced will be negligible as long as the terminal time is reasonably large.

5. APPLICATIONS

Several small models are discussed here to illustrate the sort of results which can be expected from numerical analysis of investment models. Experiments ranging from imposition of a dividend tax to an expected product price decline are considered.

5.1 Constant Returns to Scale Model

First implemented was the basic CRTS model, the second-order form of which was given in equation 4.1.6 and is repeated below:

$$K'' + K'(g-r_\tau) + \delta K(g-r_\tau-\delta) = \frac{1}{2\theta}\left((1+\gamma)(r_\tau+\delta-g) - \frac{\beta_\tau}{P_k(1-T_\tau^s)} \right).$$

Recall that g is given by:

$$g(t) = - \frac{T_\tau^{s\prime}}{1-T_\tau^s} - \frac{T_\tau^{d\prime}}{1-T_\tau^d} .$$

This model was used for several experiments regarding changes in T^d and T^s; β was held constant. Furthermore, in all experiments $T^{d\prime}$ and $T^{s\prime}$ were zero for large t; the tax and the subsidy were expected to be constant after some final change. Parameter values used in the simulations were as follows:

$r_\tau = .05,$ $\delta = .10,$ ${}_kP = 1.0,$

$\theta = 20/3,$ $\gamma = -2/3,$ ${}_\tau\beta = .25.$

In the absence of any changes, $g = 0$ and the steady state capital stock, calculated directly from the second-order form, turns out to be 1.0 for the parameters above and a zero subsidy rate. From this, the following steady state values can be calculated:

Investment $= \delta K^{ss} = .1$,
Earnings $= \beta\tau K = .25$,
Investment Cost $= P_k I(1+\gamma+\theta I) = .1$,
Dividends $= E(K)-C(I) = .15$,
Market Value $= .15/.05 = 3.0$.

Notice that γ was chosen so that adjustment costs are zero at the steady state. In each experiment, the initial capital stock was 1.0 and the model was constrained to attain the steady state at time 100. The initial dividend tax was zero. Several experiments were run including: (1) introduction of a surprise permanent dividend tax of 25% at time zero; (2) an announcement at time zero that a permanent 25% dividend tax would be implemented at time 10; 15; (4) a surprise permanent investment subsidy of 50% at time zero and (5) an announced 50% subsidy to be implemented at time 10. These experiments are discussed in detail below.

5.1.1 Surprise Permanent Dividend Tax

A surprise dividend tax expected to be permanent should have no effect on the firm's capital stock, investment, earnings, or investment cost. Dividends, however, should decrease in proportion to the tax, as should the market value of the firm. This is simply a pure profits tax which does not alter the optimal operating point and only changes the profit received. This result is obtained using the model: owners of the firm lose 25% of the value of their equity when the tax is imposed, but no other effects occur.

5.1.2 Anticipated Dividend Tax

This experiment is far more interesting because the behaviour of the firm is influenced by the announcement itself (assuming that the announcement is believed). Roughly speaking, the announcement of the tax allows owners of the firm to draw larger than usual dividends immediately prior to implementation of the tax. This occurs because the cost of investment, in foregone dividends, is much higher before the tax than after: one dollar of investment not done before the tax is one dollar of extra dividends, but one dollar of extra investment after the tax is a loss of only $.75 in dividends. Once the tax change takes effect, however, investment returns to its previous level.

The results obtained by numerical analysis are displayed in figure 5.1.2 which shows the capital stock, investment, dividends and the firm's stock undisturbed (no tax) case. Also shown are the curves corresponding to the previous experiment in which the tax was a surprise. Figure 5.1.2(a) shows the capital stock declining up to the imposition of the tax then growing asymptotically back to the steady state. The capital stock in period ten is 12.5% lower than in the undisturbed case.

It is interesting to contrast these numerical results with the analytical solution found in section 3.4.1 where the following equation for the reduction in the capital stock was derived:

$$\Delta K^1(t) = \frac{\beta_\tau (T_1^d - T_2^d)(\ \exp(-(r_\tau + 2\delta)(T-t)) - \exp(-(r_\tau + 2\delta)T)\ )}{2\theta P_k (r_\tau + \delta)(1 - T_1^s)(1 - T_1^d)(r_\tau + 2\delta)} .$$

Inserting the appropriate values and calculating the result at period ten shows that the capital stock should have fallen by .1147 units to .8853. The numerical solution yields a time ten value for K of .8850, an error of only -.03 percent. Investment, in contrast with the capital stock, declines immediately by 7.5% when the tax is announced. After the announcement investment continues to decline up to the time of implementation at which it jumps back to its original level.

The analytic results from section 3.4.1 can again be used to check numerical accuracy. Investment in the pre-implementation period is given by equation 3.4.1.2, reproduced below:

$$\Delta I^1 = \frac{1}{2\theta}\left(\frac{\beta_\tau}{P_k(1-T_1^s)(r +\delta)}\left(1+\left(\frac{T_1^d-T_2^d}{1-T_1^d}\right)\exp(-(r_\tau+\delta)(T-t)) \right) - (1+\gamma) \right).$$

Evaluating at time zero gives the true level of investment at the time of announcement; inserting the appropriate values gives $I(0)= .0930$, a reduction in the level of investment is .54 percent. The error in the drop in investment is 6.7%, somewhat larger than the error in its level. This would be even smaller if not for a peculiarity in the program used to find the solution: since the second-order form was used, the variable actually solved for was the capital stock; investment was calculated from it using a finite difference approximation and is hence less accurate. Furthermore, the initial level of investment had to be extrapolated backwards from the numerical results for the next few grid points, so the first value is particularly unreliable. More sophisticated methods could be used for this operation and the error reduced substantially.

Also important is the accuracy of the numerical solution near the time of implementation. In the true solution, investment changes discontinuously at implementation; the numerical technique approximates this with continuous function and so is inaccurate at the time of implementation. The results slightly on either side of the transition will be much more reliable. Consider investment in period nine: the analytic result is .0731 while the numerical solution is .0723. The error in the level is -1.1%, and the error in the gap is 3.0%. Presumably the percentage drop in the investment is of more interest than its level, so from the above figures the model is accurate to within about 5% of its analytic value. Further analysis of the accuracy of numerical methods and how such accuracy can be improved will be deferred to the technical paper.

Figure 5.1.2 is a graph of the dividend rate over time and clearly shows the excess of dividends in the announced-tax case over both the undisturbed case and the surprise-tax case. Dividends drop below the surprise-tax case after implementation because the capital stock is lower, a result of those discount factor gives the value of the firm, and is shown in figure 5.1.2. When the policy is announced owners of the firm lose 14.6% of the value of their equity; recall they lose 25% if the tax is a surprise. After the initial drop, the value of the firm's stock continues to decline up to the time of implementation because of the decline in the firm's capital. It is important to remember that only the initial jump hurts the equity holders because the subsequent decline in firm value is compensated by the higher dividend rate. All information is capitalized into the market value of the firm and the only possibility of earning an excess return is to possess inside information.

The results of this simple experiment with a very small model are quite striking and illustrate the importance of foresight. Because the experiment was the introduction of a dividends tax which, if it were a surprise, would have no effect at all on the firm's behaviour, all the results in the announced-tax case arise purely from the change in expectations. Furthermore, the magnitude of the effect is large enough to be of considerable significance. More detailed earnings and investment cost functions would be necessary before the results could be claimed realistic, but those presented are certainly not implausible and do provide a great deal of intuition regarding the effect of foresight on investment. In addition, the closeness of the numerical and analytical solutions shows that numerical analysis can provide a very good approximation to the true solution. This is of vital importance because many experiments of practical interest will not be analytically tractable; an example is the next experiment.

5.1.3 Temporary Dividend Tax

One interesting policy which is only slightly more complex than the announced tax is an announced temporary tax. The addition of a termination time makes analytic solution quite difficult but provides no added difficulty for numerical analysis. The policy considered here is an announcement at time zero that a 25% dividend tax will be in effect from period five to period ten. If both the imposition and removal of the tax were surprises (agents always believe the current tax will persist forever) the capital stock and investment would be undisturbed although firm value would drop in proportion to the tax while it was in effect. When agents expect the tax, their behavior is quite different, as shown in figure 5.1.3.

Announcement of the tax causes investment to drop by 8.7%, dividends to increase by 9.3%, and market value of the firm to drop by 3.7%. In the period following the announcement the capital stock declines, because of the reduction in investment, up to the time the tax takes effect. At that point dividends are sharply curtailed and investment increases. During the period the tax is in effect, owners of the firm receive their return as capital gains rather than dividends. Finally, when the tax is lifted investment drops back to its steady state level and the capital stock asymptotically approaches the steady state.

The indifference of equity owners between dividends and capital gains is obviously very important here because the optimal policy requires large swings in the dividend rate. It would be an interesting extension of the model to constrain either the dividend rate or its derivative in a manner consistent with typical corporate policy. The result should be to decrease the amplitude of variations in the capital stock and also to decrease the market value of the firm.

As suggested above, the response of the CRTS firm is quite different when the policy considered is an investment subsidy. The case of a surprise subsidy is considered in the following section.

5.1.4 Surprise Investment Subsidy

The investment subsidy considered here is a policy which has the effect of reducing the price of capital to the firm. It is not proportional to the amount of new capital goods installed, but rather to the amount spent on investment, so adjustment costs are subsidized in addition to the pure cost of new capital. This is the most realistic case, but it would be interesting to consider how a subsidy of installed capital goods would differ.

Recall the discussion of first order conditions in section 3.4 in which it was noted that, in this CRTS model, the first-order conditions could be solved sequentially. The implication was that an investment subsidy should have no effect on λ because it fails to appear in the second first-order condition. It does, however, influence investment. Recall equation 3.4.4, the investment function for this model:

$$I = \frac{1}{2\theta}\left(\frac{\lambda}{P_k(1-T_\tau^s)(1-T_\tau^d)} - (1+\gamma) \right).$$

Clearly, if a surprise subsidy is implemented, and firms believe it to be permanent, investment will immediately jump to its new steady state level. The capital stock will subsequently rise toward its new, higher steady state. Dividends will drop since the initial capital stock is too low to support both the original dividend rate and higher investment. Market value jumps up at the onset of the policy and rises as the capital stock grows. These results are shown in figure 5.1.4, and numerical results confirm intuition.

5.1.5 Announced Investment Subsidy

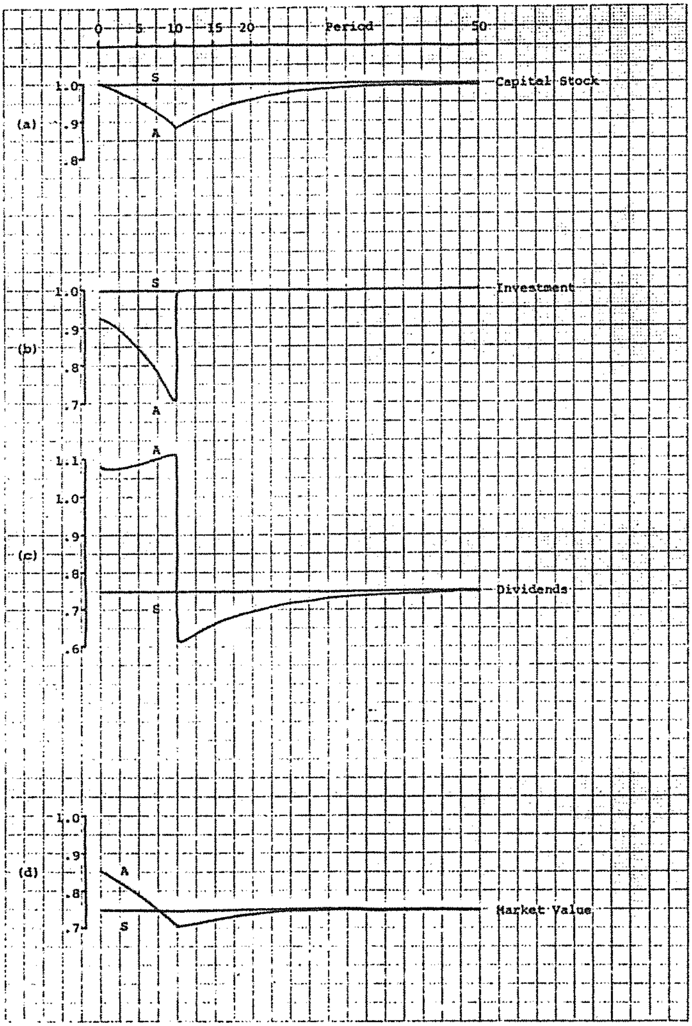
As discussed in section 3.4, advance knowledge of the subsidy does the firm no good in this model: investment and the capital stock will remain at their initial levels until the subsidy is actually implemented. The only effect of the announcement is on the value of the firm, which rises as expectations of future dividends rise. These characteristics should be observable in the numerical simulation of the experiment, and indeed are, as shown in figure 5.1.4.

Close examination reveals that before implementation, capital and investment do drift away from their initial levels by a small amount. This inaccuracy reveals one of the weaknesses of numerical analysis: because of the finite spacing of grid points used in the solution, the abruptness with which the derivative of the capital stock can change is limited. Strictly speaking, the second derivative of the capital stock is undefined at implementation; the numerical solution, however, will use a finite value determined by the spacing of the grid. The effect of this is to produce a slight rounding of sharp corners such as the one which occurs when the tax is implemented. The phenomenon is observed here but not in the dividend tax experiment because the change in the derivative at implementation is a factor of ten larger in this experiment. If the subsidy imposed is considerably smaller, 5% for example, the maximum change in the derivative of the capital stock is of the same order as in the dividend tax experiment, so the drift should be quite small. Performing this simulation generates a numerical solution which agrees with the analytic result to .01%.

Even in the 50% subsidy simulation, this drift error is not of quantitative significance because of the sharpness of the model's response. The maximum drift in investment, from the true value of 1.0, is to .974 just before implementation. At that point, however, investment jumps by 125%, so the drift as a fraction of the change in investment is only 2%. As in the dividend tax case the numerical results obtained are within about 5% of the true solution. The levels of the variables may be quite a bit more accurate, as noted in the dividends tax case. This type of inaccuracy is discussed in more detail in the technical paper, but the results shown here demonstrate that caution must be exercised when interpreting numerical results for experiments which produce optimal paths with extremely sharp corners.

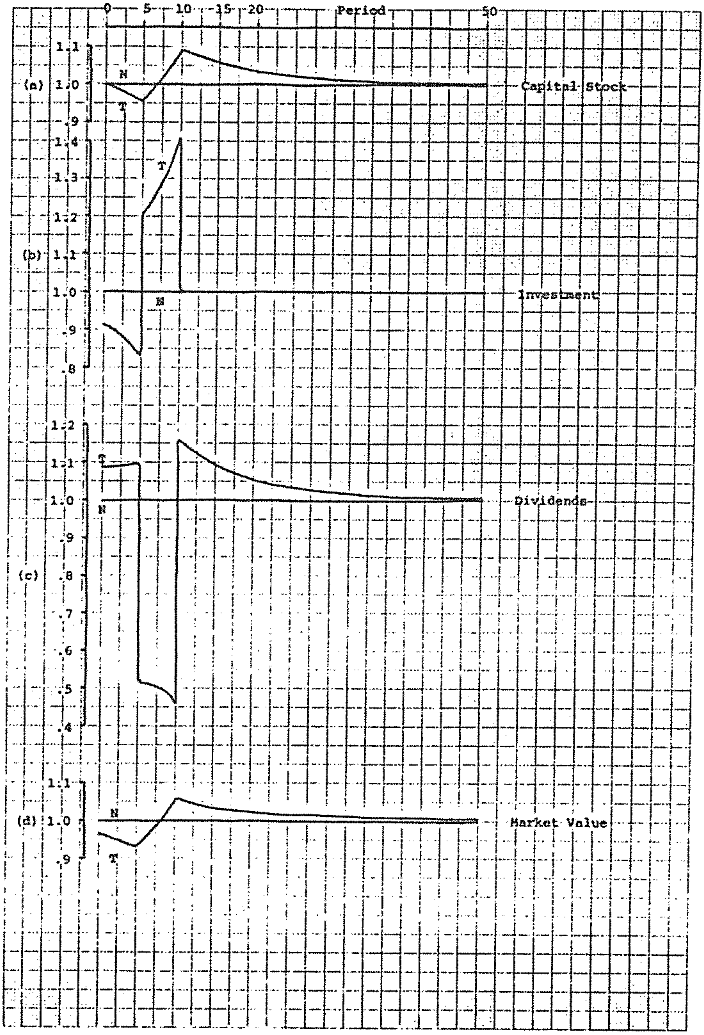
It is interesting to contrast these results with a numerical analysis of the diminishing returns model of section 3.5 since it was suggested that under diminishing returns it could be optimal to postpone a small amount of investment just before the subsidy takes effect. This is the topic of the nex section.

Figure 5.1.2: Announced and Surprise Dividend Taxes



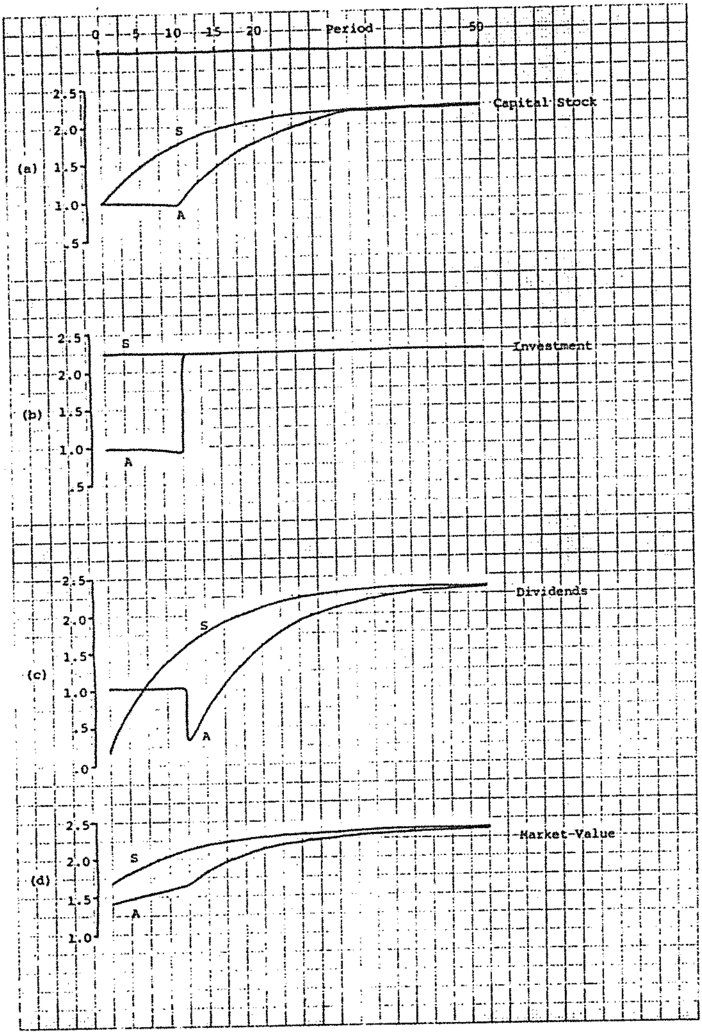
A = Announced, S = Surprise
Variables are expressed as fractions of their original values

Figure 5.1.3: A Temporary Dividend Tax



N = No Tax, T = Temporary Tax
Variables are expressed as fractions of their original values

Figure 5.1.4: Announced and Surprise Investment Subsidies



A = Announced, S = Surprise
Variables are expressed as fractions of their original values

5.2 Diminishing Returns Model

The second problem analyzed was the diminishing returns model described in section 3.5. The earnings function has already been discussed; the investment cost function used was:

$$C(I) = \theta P_k I^2 (1-T^s_\tau).$$

This is a special case of the previous function: the parameter γ has been set to -1.0 for algebraic simplicity. Applying the general first-order conditions from section 3.3, with the appropriate modifications for the current earnings and cost functions, and converting to second-order form in the state variable K gives:

$$K'' + K'(g-r_\tau) + \delta K(g-r_\tau-\delta - (\frac{\alpha}{2\theta\delta P_k(1-T^s_\tau)})) = - \frac{\beta_\tau}{2\theta P_k(1-T^s_\tau)} .$$

The term $\alpha/2\theta\delta P_k(1-T^s)$ is the change in marginal earnings over the change in marginal investment cost for an extra unit of capital: it is the rate at which marginal earnings are diminished with increasing investment.

This model was used for a number of experiments involving changes in T^s; β and r were held constant. Parameter values used in the simulations were as follows:

$r_\tau = .05,$ $\delta = .10,$ ${}_kP = 1.0,$

$\theta = 20/3,$ $\alpha = .10,$ ${}_\tau\beta = .30.$

The steady state of the undisturbed model can be calculated by applying the second-order equation with K'', K' and g set to zero; if the subsidy is zero, the above parameters yield a steady state of 1.0. The following values hold at the steady state:

Investment $= \delta K^{ss} = .1$,

Earnings $= \beta\tau K^{ss} - (\alpha/2)K^{ss2} = .25$,

Investment Cost $= \theta PkI^2 = .06$,

Dividends $= E(K) - C(I) = .183$,

Market Value $= .183/.05 = 3.667$.

In each experiment the original capital stock was 1.0 and the new steady state was imposed at time 100. Two main experiments were run: (1) a surprise permanent subsidy of 50%, implemented at time zero, and (2) an announced subsidy to take effect at time ten. In addition, the results of the announced-subsidy experiment were examined for sensitivity to α and the size of the subsidy, but discussion of this will be postponed to the appendix.

5.2.1 Surprise Permanent Subsidy

The main effect of a permanent subsidy is to increase the steady state capital stock in the industry. Because of adjustment costs, the firm will be unable to change from the old steady state to the new immediately; it will only approach the new level asymptotically. Furthermore, since the model no longer has CRTS, investment will not jump immediately to the new steady state level, as it did in the previous model. Actually, it jumps somewhat higher than the new steady state initially and declines toward it over time. This occurs because the overall effect of the subsidy is to increase the steady state capital stock in the industry, and since the firm faces diminishing returns to capital the marginal earnings of capital are higher in the early periods when the firm has a small amount of it. That is, at low levels of capital the returns to an additional unit are higher than at high levels of capital, so investment is pushed higher in the early periods than at the steady state. This conclusion is borne out by numerical analysis, which indicates that the steady state level of capital jumps by 50% (as does the steady state level of investment), but investment in the initial periods jumps by an additional 26% to 76% of its original value. The consequence of early high investment is a depression in dividends (since the firm's earnings are unchanged), and this too is observed. Finally, owners of the firm's equity receive a 29% windfall capital gain at announcement after which the value of the firm rises asymptotically toward the steady state.

Although the steady state capital stock rises by 50%, dividends and firm value rise by only 43% because of the nonlinearity of the earnings and investment cost functions. The effect of the subsidy on the model is as follows: (1) the steady state capital stock jumps by an amount which can be determined from the second-order equation, (2) steady state investment is increased by the same percentage, and (3) earnings and investment costs both rise, changing dividends and market value. That in this experiment the steady state capital stock increases by the same percentage as the subsidy is a numerical accident, as can be seen by examining the second-order equation.

This experiment serves to illustrate the difference in behaviour arising from relaxing the CRTS assumption of the first model. Results for this experiment and the one which follows are shown in figure 5.2.1.

5.2.2 Announced Subsidy

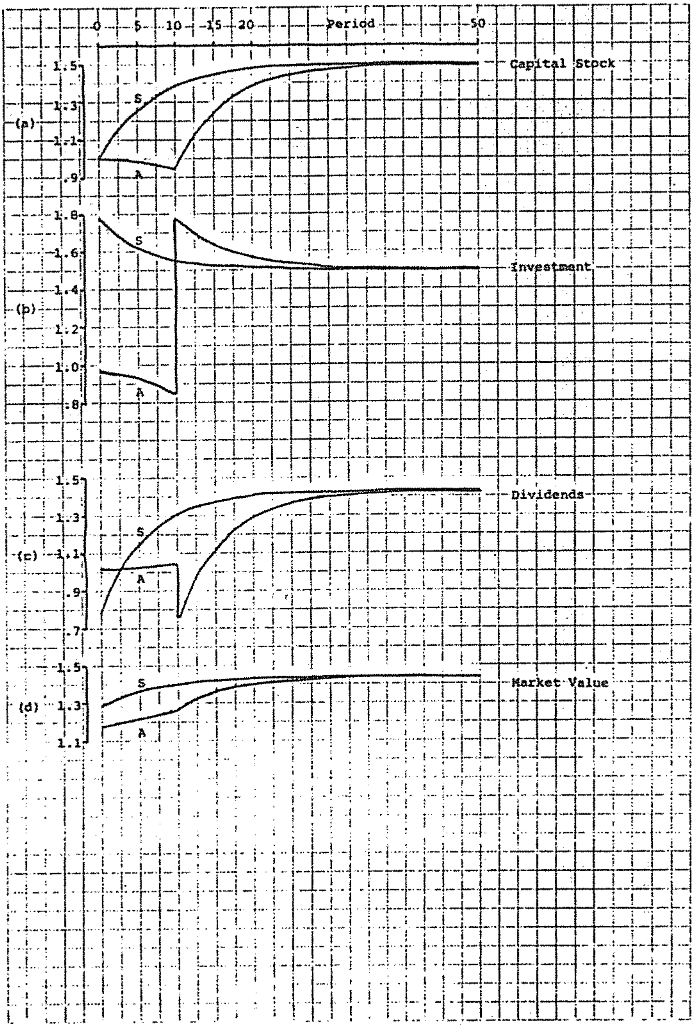
In contrast to the CRTS case, advance knowledge of a capital subsidy does influence the firm's behaviour, at least to a small extent. The reasoning is as follows. Before the subsidy is implemented the firm faces a decision in which paying extra dividends now reduces its capital stock and hence its market value at implementation. When the policy takes effect, this lower capital stock will prompt higher investment than if the firm had not paid the extra dividends because of the high marginal earnings of capital. Thus, in this model, pre-implementation dividends increase post-implementation investment; in contrast, post-implementation investment independent of the capital stock in the CRTS model. The firm's value at implementation is lowered by pre-implementation dividends to the extent that its capital stock at implementation is diminished relative to what it otherwise would have been. The acceleration in investment in this model, however, will eliminate the difference in the capital stock in the first few periods. For this reason, a drop in capital at implementation reduces the value of the firm less than in the CRTS case, and it is optimal to postpone a small amount of investment before the subsidy in favour of extra dividends. Numerical analysis bears this out, as shown in figure 5.2.1.

The announcement results in a drop of 5% in the capital stock at implementation, but part of this is due to the numerical inaccuracy discussed above (in the appendix it will be argued that this result is significant in the sense that the error generated is less than 5%). Because the subsidy is not to be implemented until period ten, owners at announcement receive a windfall gain of 18% rather than the 29% gain in the surprise case. Since the capital stock is lower, the value of the firm at implementation is lower than at implementation of the surprise subsidy: 27% higher than the reference case instead of 29%.

Overall, the effect of diminishing returns is to accelerate the approach to the steady state. This occurs because the return to extra capital is high at low levels of capital while the cost of investment is independent of the level of the capital stock. A given amount of investment costs the same at high and low levels of capital, but yields higher benefits in the latter. The effect of this acceleration is to reduce the cost of low levels of capital at implementation, allowing excess dividends to be drawn.

Having investigated the effect of diminishing returns, it is useful to turn once more to the constant returns case. In the next section, an extended constant returns model is developed which has a more realistic investment cost function and is suitable for more kinds of policy experiments.

Figure 5.2.1: Investment Subsidies in the Diminishing Returns Model



A = Announced, S = Surprise
Variables are expressed as fractions of their original values

5.3 Extended Constant Returns Model

This was constructed along the lines of the simple CRTS model, but with a number of modifications to allow it to be used for a wider variety of simulations. The earnings function was derived from short run profit maximization subject to a Cobb-Douglas production function, as discussed in section 3.1:

$$E(K) = \left(\frac{1 - \varepsilon}{\varepsilon} \right)\left(\frac{\varepsilon p}{w} \right)^{1/1-\varepsilon} wK. \tag{5.3.1}$$

The function possesses CRTS in K and is homogeneous in wages and prices. In contrast to the previous CRTS model, agents will now have expectations about wages and prices rather than just the overall marginal earnings of capital.

The investment cost function was slightly different also. Here it was assumed that the firm produces its own installed capital out of raw capital goods and labour. The production function was taken to be:

$$I = \min \quad \{ x_k \, , \, (L_i/\theta)^{.5} \},$$

where x_k is raw capital, Li is labor on installation and θ is a parameter. This particular form was chosen because it generates an investment cost function similar to that used in the simple CRTS model but with greater detail. Finished investment goods are thus linear in raw goods and quadratic in labour, so adjustment costs will enter through the labour requirement. The corresponding cost function is:

$$C = (P_k x_k + wL_i)(1-T^s_\tau).$$

Minimizing investment costs subject to the specified investment goods production function yields the following investment cost function:

$$C(I) = (P_k I + w\theta I^2)(1-T^s_\tau).$$

Applying the general first-order conditions from section 3.3 gives the following:

$$\lambda = (\ P_k + 2w\theta I\ )(1-T_\tau^s)(1-T_\tau^d),$$

$$\lambda' = (r_\tau + \delta)\lambda - \left(\ \left(\frac{1-\varepsilon}{\varepsilon}\right)\left(\frac{\varepsilon p}{w}\right)^{1/1-\varepsilon} w\right)(1-T_\tau^d),$$

$$K' = I - \delta K.$$

Finally, conversion to second-order form gives the equation below:

$$K'' + K'\left(g - r_\tau + \frac{w'}{w}\right) + \delta K\left(g - r_\tau - \delta + \frac{w'}{w}\right) = \frac{P_k}{2w\theta}\left((r_\tau + \delta - g) - \frac{P_k'}{P_k} - \frac{(dE/dK)}{P_k(1-T_\tau^s)}\right),$$

where

$$\frac{dE}{dK} = \left(\ \frac{1-\varepsilon}{\varepsilon}\ \right)\left(\ \frac{\varepsilon p}{w}\ \right)^{1/1-\varepsilon} w\ .$$

When P_k and w are constant this reduces to the basic CRTS model with $\gamma = 0$ and θ replaced with $w\theta$. The following parameters were used:

$r_\tau = .05,$ $\delta = .10,$ ${}_{k}P = 1.0,$

$\theta = 10/3,$ $\varepsilon = .5.$

The reference case had $P_k = w = p = 1.0$ and no dividend tax or investment subsidy. Evaluating dE/dK shows the initial marginal earnings of capital to be .25, and the steady state can be found in the usual way from the second-order equation. In the absence of policy shocks, the steady state capital stock is 1.0 and the following values hold:

Investment $= \delta K^{ss} = .1,$

Earnings $= (dE/dK)K = .25,$

Investment Cost $= P_k I + \theta w I^2 = .133,$

Dividends $= E(K) - C(I) = .117,$

Market Value $= .117/.05 = 2.333.$

Once again, each experiment was begun with the capital stock at the steady state and the model was constrained to achieve the new steady state at time 100. Three groups of simulations were used: (1) a drop in the product price, (2) a increase in wages, and (3) an increase in the price of capital goods. For each group, several simulations were run to determine the effect of an announced change relative to a surprise.

5.3.1 Surprise Price Decline

This experiment is best thought of as the problem faced by an import-competing firm when tariffs are to be reduced, but it also applies to an industry about to experience a drop in demand as its product becomes obsolete. Considered in this section is the response of investment to a surprise decline in product price of 10%; the role of anticipation will be discussed later.

The primary effect of a drop in price is to cause the industry to shrink by reducing investment. If the price change is unanticipated, at implementation investment drops immediately to its new steady state level for the reasons discussed above. The final capital stock can be calculated in the usual way and it turns out to be 0.525. The optimal amount of capital falls by more than 10% for two reasons: the earnings function is not linear in the output price, as shown in equation 5.3.1, and the steady state is not linear in earnings because of the investment cost function; both effects tend to lower the optimal capital stock in the event of a price decline.

Because of the CRTS earnings function, investment drops immediately to .0525; dividends jump up initially and then decline asymptotically toward the new steady state. The owners of the firm suffer a windfall capital loss of

34%, after which the firm's value declines with its declining capital stock. These results, along with those for the following experiment, are shown in figure 5.3.2.

5.3.2 Anticipated Price Decline

The previous policy is a disaster for owners of equity and leads to the question of whether announcing the change in advance would diminish this loss by allowing the firm to change its behaviour prior to implementation. In practice, an announcement allows the firm to pay high dividends in the early period before the price decline and this will help to maintain the firm's value. The numerical results obtained confirm this, as shown in figure 5.3.2.

Investment drops abruptly when the price change is announced, but only to .0985 instead .0525. After the initial drop it continues to decline until it is exactly .0525 when the new policy takes effect. Investment is kept above its long-run steady state level by the high earnings of capital before the price drop, but it is optimal to cut back on investment to an extent to pay higher dividends. The dividend rate is relatively low at first because of the long interval on which earnings accrue to capital before the price drop. Later, when little additional high-price earnings are to be had from additional investment, the dividend rate rises.

All of this is reflected in the value of the firm, as shown in figure 5.3.2. Windfall losses are limited to 20% when the policy is announced in advance, a considerable improvement over the 34% loss due to a surprise price decline. Furthermore, at period ten the capital stock in the announced-policy case is .82, compared to .70 for the surprise policy. This is important because the putative goal of eliminating a tariff is to reduce excessive capital in an industry. The example here shows that announcing the removal of a tariff in advance substantially reduces the loss to equity owners but eliminates excess capital more slowly than in the surprise case. This may in itself be of some benefit if there is a large amount of firm-specific labour involved in production: such labour is eliminated more gradually when the policy is announced.

Thus the effects of announcing the policy in advance are: (1) to limit windfall losses to current owners, (2) to hold investment higher in early periods and (3) to slow elimination of capital from the industry. Furthermore, the length of the pre-implementation period could be chosen to optimize policy goals: short to eliminate capacity rapidly at the expense of current owners and workers; longer to reduce the burden on owners and employees but with slower elimination of capacity. Notice that the difference rests entirely on the assumption that agents believe the announcement. If they don't believe it, all the consequences of the policy will be exactly as it was a surprise. This is examined further in the next section.

5.3.3 Price Decline in Stages

If it is difficult for the government to make credible an announcement that it will implement a policy far in the future, it may be helpful to conduct it using a series of small steps. As an alternative to the announced period ten reduction considered above, the government could decrease prices to 95% in period five and then to 90% in period ten. If this policy is believed, all variables follow paths intermediate between the surprise and announced one-step reductions. All three policies are shown in figure 5.3.3. If agents are skeptical of the government and only believe policies which have been partly carried out, this two stage price decline will establish credibility in period five. The response of the model will be as though a surprise 5% decline had occurred in period five in combination with an announced additional decline five periods in the future. Under this policy owners of capital suffer only a 30% loss at period five, rather than the 34% loss from the one-step surprise policy. Clearly the number and timing of steps provides an additional tool to be used to achieve policy goals, particularly when implementation of some parts of the programme helps establish credibility for the rest of it. This aspect should be studied in more detail by allowing agents to have expectations with non-trivial distributions instead of requiring that they believe specific parameters will occur with probability one.

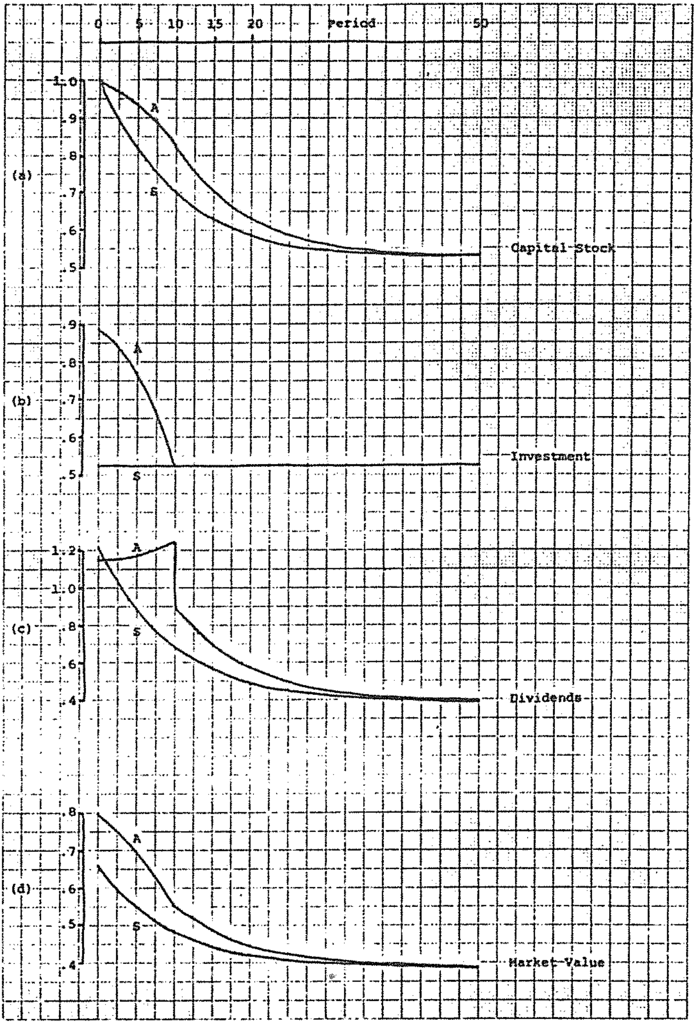
5.3.4 Wage Increase

The results for a surprise wage increase of 10% and an announced increase for period ten are shown in figure 5.3.4. Qualitatively, the outcome is much like the price decline experiment, although wages enter the investment cost function in addition to affecting the marginal earnings of capital. Announcing the change in advance limits windfall losses to about 12%, instead of almost 20%.

5.3.5 Price of Capital Increase

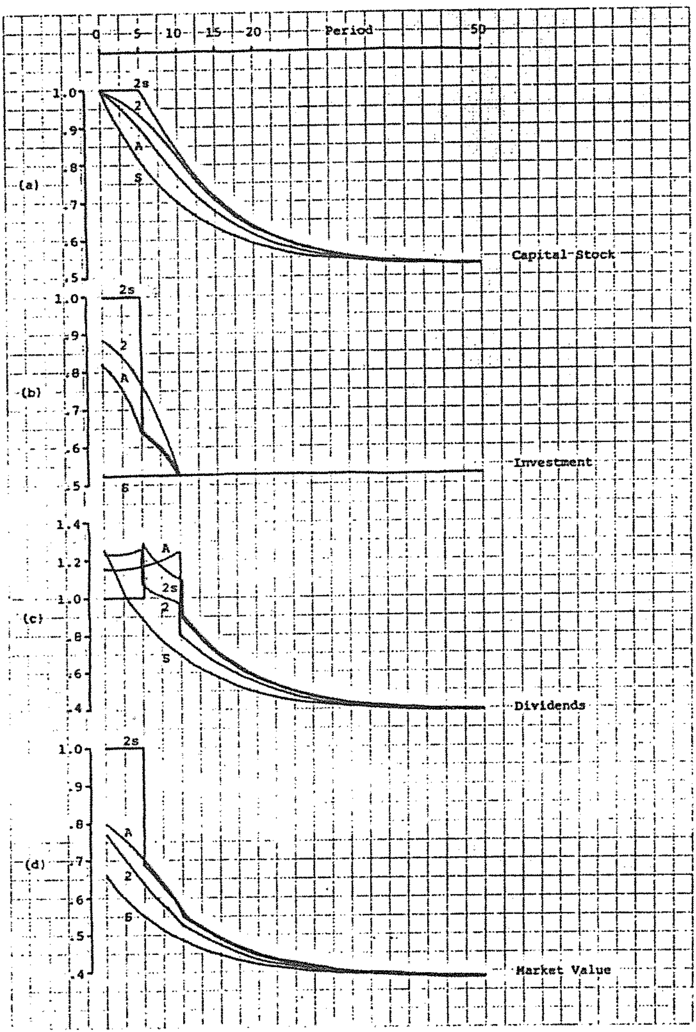
The outcome of both a surprise and an anticipated 10% increase in the price of capital is shown in figure 5.3.5. Qualitatively, this should be just the inverse of the capital subsidy programme discussed in section 5.1.2, and examination of the appropriate graphs confirms this. For the reasons suggested above the firm is unable to exploit its advance knowledge because the returns to investment are independent of the capital stock.

Figure 5.3.2: Surprise and Anticipated Product Price Decline



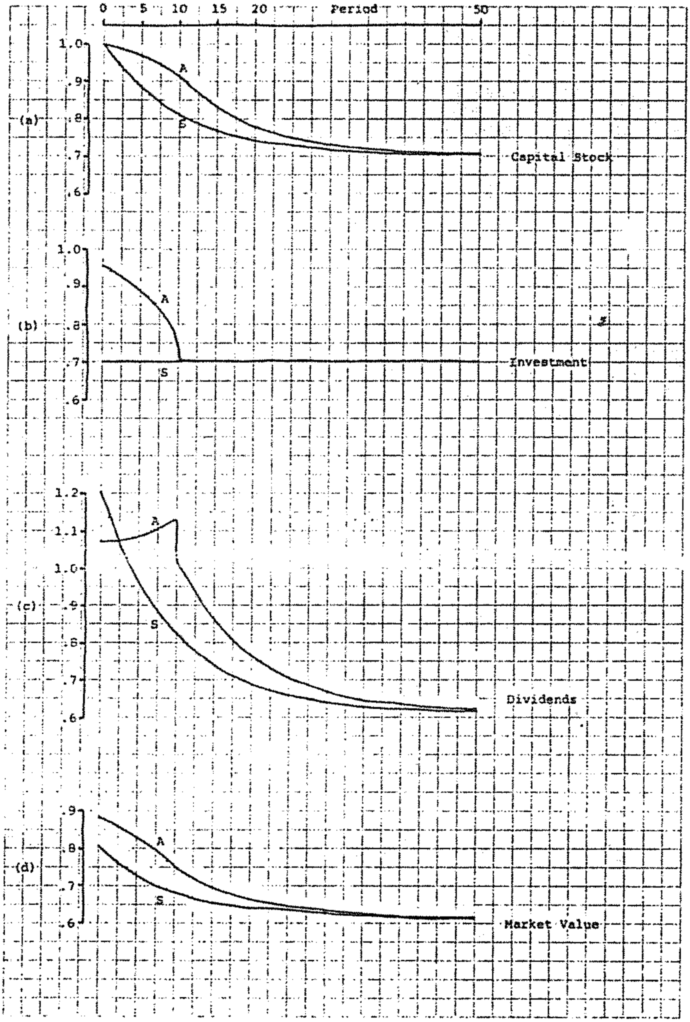
A = Anticipated, S = Surprise
Variables are expressed as fractions of their original values

Figure 5.3.3: Price Decline in a Number of Stages



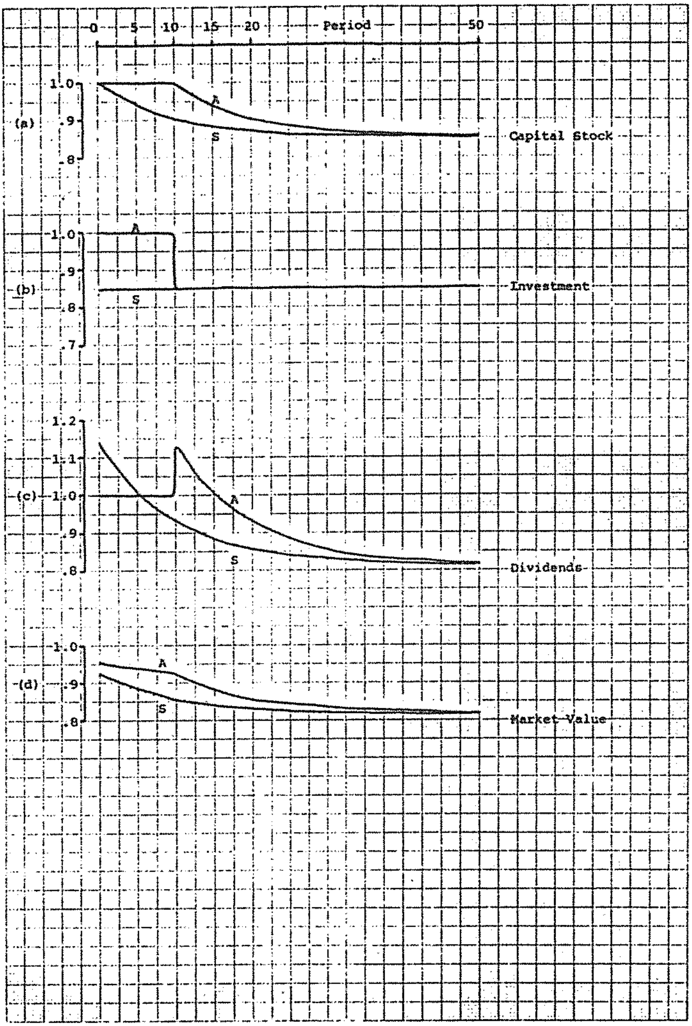
A = Announced one-step, S = Surprise one-step,
2 = Announced two-step, 2s = Two-step, surprise at period 5
Variables are expressed as fractions of their original values

Figure 5.3.4: Wage Increase



A = Announced, S = Surprise
Variables are expressed as fractions of their original values

Figure 5.3.5: Price of Capital Increase



A = Announced, S = Surprise
Variables are expressed as fractions of their original values

6. EXTENSIONS

The models presented here are all fairly simple in structure, and a number of extensions are possible which would make them more realistic and broaden the class of problems to which they could be applied. One example is making agent's expectations of exogenous variables more complex than just complete confidence in government proclamations. Another, albeit obvious, improvement would be to use more realistic earnings and investment cost functions. This could be done fairly easily, and the analysis of the general case in section 3.3 would still apply. Ideally, these functions would be estimated econometrically.

An important extension which can be readily introduced is bond finance; an interesting model could be built in which the firm's leverage ratio was a control variable in the value maximization problem. Summers (1981) has built a model which includes bond finance and a great amount of detail regarding the tax structure, although the leverage ratio is fixed. Also, the number of shares of stock outstanding was held constant in these models, but it could be considered a control variable.

Disinvestment would be an interesting extension, as mentioned in the text. Although it would be inappropriate in a CRTS model, it could be introduced with diminishing returns. Related to this is the constraint that investment be non-negative, which was satisfied in all of the current models because each experiment resulted in an interior solution (positive gross investment). Even without the possibility of disinvestment, it would be of interest to examine cases where optimal investment is zero for some period of time.

One major extension would be to link a series of solutions together over time. This would allow the information set to evolve resulting in a succession of optimal solutions to the planning problem. The actual path of the capital stock would be an envelope of these individual solutions. A preliminary effort has been made in this direction and no substantial difficulty appears to exist.

6.1 Homogeneous Adjustment Cost Function

As mentioned above, the basic adjustment cost function used here is not homogeneous in investment and capital. This means that the results obtained will depend on the scale of the economy, a property which will be undesirable in some models. It is not difficult to add homogeneity, but it can only be done at the cost of eliminating linearity of the first-order conditions. This will make an analytical solution usually impossible and will complicate numerical analysis. In this section, such a model will be developed to illustrate these difficulties. Several solution techniques will also be suggested.

It is convenient to change notation from that used previously: in this section the firm's control variable will be the investment rate rather than the level of investment. The two are related by I = iK, where I is the level of investment and i is the investment rate. With this change the firm's problem takes the form:

$$\max \int_t^{\infty} (E_\tau(K) - C_\tau(i,K))(1-T_\tau^d)\exp(-r_\tau(u-t))du,$$

$$\text{subject to} \quad K^{\cdot} = iK - \delta K.$$

Solving this using the current value multiplier results in the following first-order conditions:

$$\lambda = \left(\frac{\partial C_\tau}{\partial i} \right)\left(\frac{1}{K} \right)(1-T_\tau^d),$$

$$\lambda' = (r_\tau + \delta - i)\lambda - \left(\frac{\partial E_\tau}{\partial K} - \frac{\partial C_\tau}{\partial K} \right)(1 - T_\tau^d),$$

$$K' = iK - \delta K.$$

To derive the final form of the first-order conditions, all that is needed is to specify appropriate earnings and investment cost functions. The earnings function used here will be the constant returns version discussed above, but investment costs will now be given by:

$$C(i,K) = P_k Ki(\ 1 + \phi(i)\ )(1 - T_\tau^s),$$

where the adjustment cost function is:

$$\phi(i) = \gamma + \theta i,$$

so the overall investment cost function is:

$$C(i,K) = P_k Ki(\ 1 + \gamma + \theta i\ ).$$

Inserting the partial derivatives of these functions into the first-order conditions above gives the following system:

$$\lambda = P_k(\ 1 + \gamma + 2\theta i\ )(1 - T_\tau^s)(1 - T_\tau^d),$$

$$\lambda' = (r_\tau + \delta - i)\lambda - (\beta_\tau - P_k i(1 + \gamma + \theta i)(1 - T_\tau^s))(1 - T_\tau^d),$$

$$K' = iK - \delta K.$$

This is equivalent to the second-order equation:

$$i' + i\left(g - r_\tau - \delta + \frac{P_k'}{P_k}\right) + \frac{i^2}{2} = \left(\frac{1+\gamma}{2\theta}\right)\left(r_\tau + \delta - g - \frac{P_k'}{P_k}\right) - \frac{\beta_\tau}{2\theta P_k(1 - T_\tau^s)} .$$

The second-order character of this relationship is obscured because it has been expressed in terms of the investment rate; using the constraint to eliminate investment in favour of capital will produce the usual second-order form.

Because the system is nonlinear, it is not possible to solve it using the analytical techniques described above. Numerical methods, finite differences in particular, will still be useful, but are more cumbersome because of the nonlinearity of the model. One way to proceed is to linearize the model and solve it in that form, but this results in a certain amount of inaccuracy. Further discussion of this, and a description of how the linearization could be done, appears in the technical paper.

In summary, introduction of a homogeneous adjustment cost function can be accomplished but at a high cost in terms of tractability. For this reason, the models here used simpler adjustment cost functions so analytical results could be obtained. In small models used to develop intuition or to represent a single firm or industry, this is probably not a serious drawback, but it would be important in a model of economy-wide investment.

7. CONCLUSIONS

The most important result obtained here is that foresight matters a great deal. This was vividly demonstrated by the CRTS model and the dividend tax experiment, in which an extremely simple model and experiment generated a major difference between a surprise and an announced policy. Not only was the capital stock different at implementation, but investment and the firm's market value changed substantially at the time of announcement. It is impossible to ignore the effect foresight, even in a short run model.

Not only are such models useful for understanding the path of investment and capital, but they also provide information on the capitalization of policies into the market value of the firm. This is of vital importance if the welfare of equity owners could influence the decision. Even when the decision maker is not concerned with their welfare in itself, the possible windfall gains or losses will provoke a considerable amount of political pressure for or against the policy.

Overall, introducing foresight provides a large amount of insight and flexibility at a small analytic cost. It extends the range of policy questions which can be considered, and it provides detail on windfall distributional effects which are not otherwise available. Finally, it differentiates between surprise and anticipated policies and, under reasonable assumptions, the difference between these can be considerable. The only disadvantage to the use of sophisticated foresight models is that they will usually require numerical analysis to solve. As demonstrated above, however, techniques are available to perform such analysis which produce highly accurate results at minimal computing cost.

APPENDIX

A. Isolating the Announcement Effect

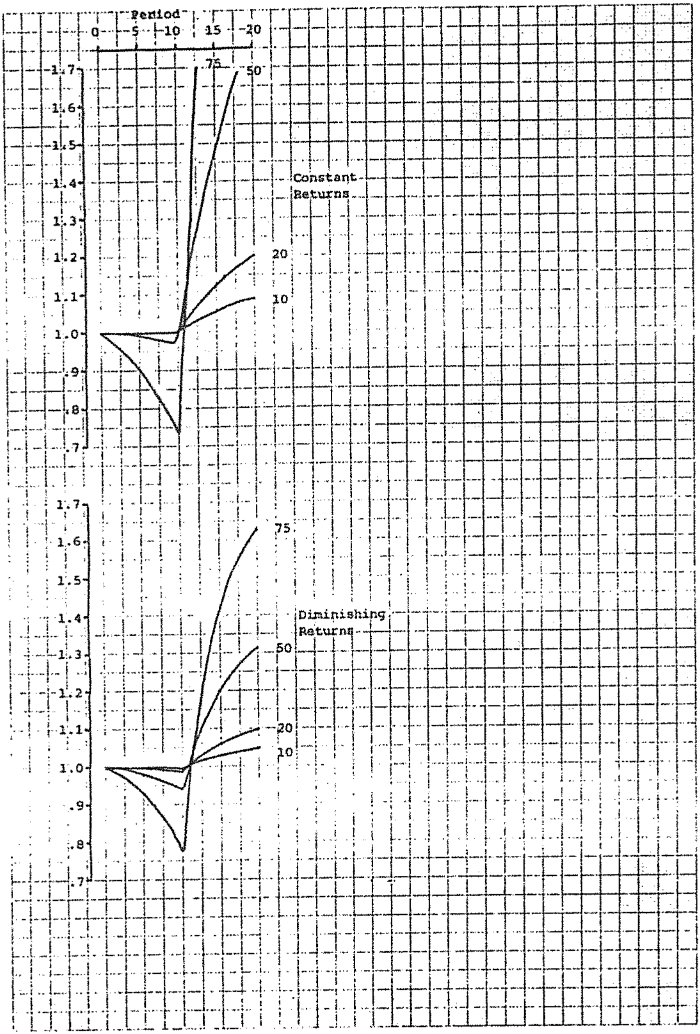
Numerical error near corners in the path of capital arises because of discontinuities in the second derivative of capital: the derivative goes abruptly from zero to a large positive value. This suggests the error should depend only on the derivative of the optimal path of the capital stock; the structure of the model matters only to the extent that it affects this derivative. Consequently, the error in the diminishing returns model near implementation of the subsidy should be similar to that in the CRTS model, if the post-implementation K' is similar. Also, the true diminishing returns solution should be sensitive to the value of α, the coefficient of the quadratic term in the earnings function. The first effect can be examined by observing the behaviour of the two models as the size of the subsidy is varied, since one effect of this is to change the derivative of capital after implementation. Both models were subjected to announced subsidies of 10, 20, 50 and 75%, and the results for the first 20 periods are shown in figure A.1. The derivatives at time 10.25 were as shown:

Subsidy (%)	Constant Returns	Diminishing Returns
.10	.0134	.0091
.20	.0303	.0200
.50	.1244	.0741
.75	.3936	.2019

The obvious conclusion from observing the CRTS results is that the solution technique breaks down for very large shocks, although it performs as expected for small changes. As discussed in the technical paper, this could be remedied by using more grid points in the finite difference approximation. For any particular subsidy, the CRTS solution has a substantially higher value of K' immediately after implementation, so the error in it is an upper bound on the error in the diminishing returns model. The conclusion to be drawn is that the 6% decline in capital at implementation generated by the diminishing returns model is overstated, but only by as much as 2.5%.

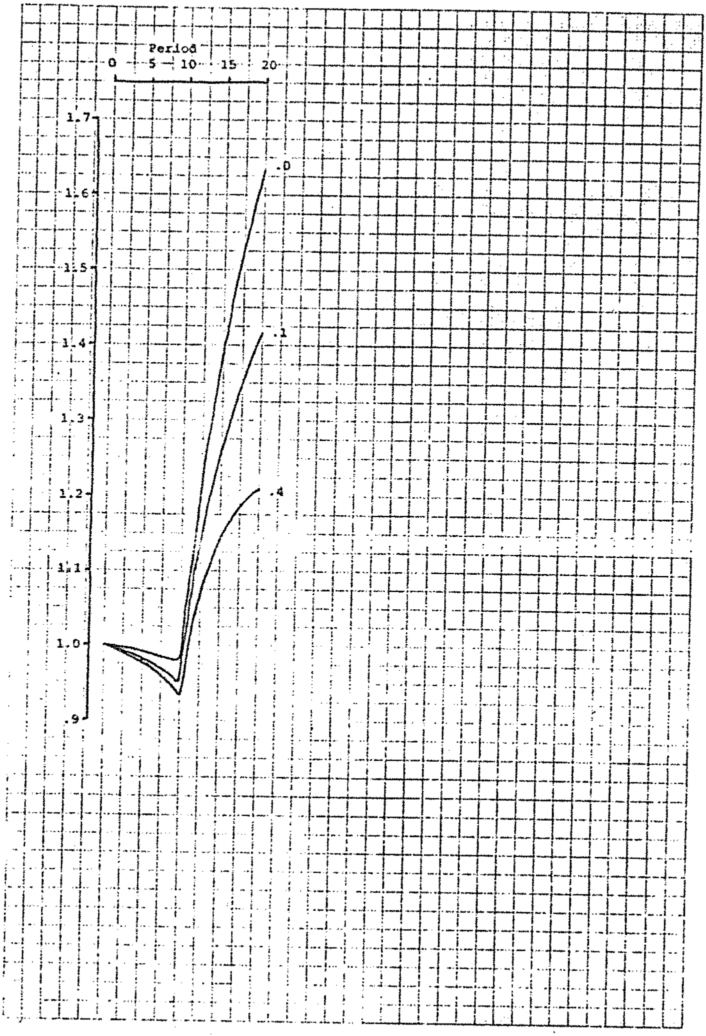
Further evidence that the diminishing returns model is behaving as expected is given by its response to variations in α. The results of the announced 50% subsidy experiment are graphed in figure A.2 for a number of values of α. Clearly, increasing α leads to a decrease in capital at implementation. This occurs because as α increases, the difference between post-implementation investment and steady state investment increases. The capital stock gets to the steady state more rapidly, so the penalty for having low capital at implementation is less. Also, since the derivative of the capital stock is decreasing in α (because the steady state is decreasing), the numerical error should also be smaller for larger α. The gap in the $\alpha = 0$ schedule is thus an upper limit of the error in the other curves.

Figure A.1: Influence of Subsidy Size on Numerical Accuracy



Curves shown are the path of the capital stock for a subsidy of the indicated percentage. The capital stock is expressed as a fraction of its original value.

Figure A.2: Effect of α on Numerical Accuracy



The curves shown are the path of the capital stock (expressed as a fraction of its original value) for the indicated values of α.

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