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A DYNAMIC MODEL OF A JOINT FIRM-HOUSEHOLD*

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I. INTRODUCTION

In the existing literature, the two major areas of empirical application of dynamic models are the theory of investment and the theory of consumption. The majority of investment models take as given that the criterion function is the present value of the firm. This assumption can be traced back to Fisher [7], who pointed out that, under the assumption of a perfect capital market, consumption and investment decisions can be separated. The firm acts to maximize its present value, and the consumer takes this present value as given to maximize utility. In the investment literature we should mention Arrow [1], Jorgenson [12], Eisner and Strotz [6], Treadway [22, 23], Lucas [17] and Gould [8], and a variation of the criterion function by Wong [25]. In consumption theory, there are a number of theoretical models, such as Tintner [20], and recently Lluch [14, 15, 16] has developed an empirically implementable model of intertemporal consumption theory, called the Extended Linear Expenditure System (E.L.E.S.). Essentially Lluch takes the present value of the household as given, in the Fisherian tradition.

If we are interested in a joint firm household entity, for example a farm, where a single decision-making unit carries out decisions on consumption, production, investment and financing, the initial temptation is to appeal to Fisher's suggestion, treat the productive entity first to maximize present value, use this present value to determine consumption behaviour, and hence to simply combine the above models. But the assumption of a perfect capital market, a pre-requisite for such an approach, appears to be a critical violation of reality in many areas. The approach taken in this paper is to assume an imperfect capital market, and treat the consumption-investment-production-financing decisions as interdependent.

To allow concentration on long-term changes, the model is specified as a free-end-point type—at each point in time the enterprise is allowed to reorganize its portfolio of capital assets to give an optimal configuration. Two points about this assumption need to be made. First, application is aimed at the micro level so that supply of capital goods (even land) to each individual enterprise is assumed infinitely elastic and general equilibrium considerations are avoided. This is in contrast to Vincent, Dixon and Powell [24] where aggregative data is used, and such problems are overcome by holding capital assets fixed, thereby avoiding the intertemporal aspect. Secondly, however, general equilibrium

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aspects cannot be ignored completely, since otherwise a single model would predict only one pattern of production, clearly at variance with reality. This problem is overcome by considering broad classes of models based on broad categories of activities and the corresponding production functions. Thus, for example, a dairy farmer is constrained to face a dairying-type production function, and so will not suddenly change to wheat if wheat prices rise. In effect, supply aspects are introduced by differentiating between production functions. These constraints are technical in nature, and best imposed from outside the model. Thus the apparent ease of re-allocation introduced by the free-endpoint assumption is in fact constrained within the model by the prior specification of the production function.

The framework of Lluch's E.L.E.S. is taken as the basic starting point, and E.L.E.S. is reformulated in a way to allow further extensions in section II. We use duality theory to simplify Lluch's model, arguing that this is the "natural" approach. We also introduce the idea of synthesizing a closed loop control and discuss its econometric implications, which is not considered by Lluch. In section III the natural (dual) specification of the production sector is formulated, and the financial side is introduced in section IV, where the average rate of interest is assumed to be an increasing function of the debt-asset ratio. For want of a better term, we refer to this specification as an imperfect capital market. The complete model is presented in section V, with a closed form solution derived under one set of functional restrictions. In section VI we draw an important distinction between behaviour in planning time and behaviour in calendar time, while section VII contains some further comments on empirical implementation.

II. THE EXTENDED LINEAR EXPENDITURE SYSTEM

The Extended Linear Expenditure System is introduced by Lluch [14], and further developed in a number of other papers [15, 16]. This model may be clarified and made more accessible by drawing on the simplifications suggested by modern duality theory (see, for example, Diewert [4]) and by casting it in a framework more consistent with modern control theory (see, for example, Arrow [2], Hestenes [10], Intriligator [11], Pontryagin *et al.* [19]). Such a reformulation may lead to further generalizations, and allows incorporation of suggestions on the appropriate budget constraint by Arrow [3] and Klijn [13].

It is of some interest to note that E.L.E.S. is an example of a control problem in which synthesis is possible; that is, a closed-loop feed-back control law expressing the optimal control at each point in time as a function of the state at that time can be derived explicitly. Such "well-behaved" control problems seem to be very rare, and the majority of control theory references quote only the quadratic criterion, linear control law case as an example in which synthesis is possible. (A well-known example from economics which falls into this category is the Eisner-Strotz [6] investment function.) Econometrically, closed-loop control relations are important because it is these relations which are stable across samples and through time. Incidentally, closed-loop control laws ensure consistency in the sense of Strotz [21], which is not so obvious in the case of "initial period" controls. In the case of E.L.E.S., however, consistency is not a problem, by a result due to Strotz.

Following Lluch, define the following set of variables:

$q(t)$: n-vector of consumption good flows, $q > 0$,
p : n-vector of prices corresponding to q,
$y(t)$: exogenous flow of labour income,
M : initial money holdings,
δ : subjective rate of time discount,
ρ : rate of interest in a perfect capital market,
U : instantaneous utility function, defined on q
E : total consumption expenditure $= p'q$

Further, define the consumers net worth at time zero as

$$W^0 = \int_0^\infty e^{-\rho t} y(t)\, dt + M \tag{1}$$

and his indirect utility function by

$$V[E,p] = \max \{U(q) : p'q \leqslant E\}. \tag{2}$$

Then the consumption function is the solution E^* to the problem:

$$\max_{E(t)} \int_0^\infty e^{-\delta t} V[E(t), p]\, dt \tag{3}$$

subject to

$$\dot{W} = \rho W - E$$
$$W(0) = W^0$$
$$\lim_{t\to\infty} W(t) \geqslant 0.$$

The corresponding demand system is derived by applying Roy's Identity to V and evaluating at E^*:

$$q_i = -\frac{\partial V[E^*,p]}{\partial p_i} \bigg/ \frac{\partial V[E^*,p]}{\partial E} \tag{4}$$

By an open-loop solution is meant a solution for E as a function of t and W^0, whereas a closed-loop solution specifies E as a function of $W(t)$. Substitution in (4) gives the corresponding open and closed-loop solutions for the q_i, and multiplication through by p_i gives an expenditure system.

To generate E.L.E.S. let U be Klein-Rubin:

$$U(q) = \Sigma\beta_i ln\,(q_i - \gamma_i);$$

the indirect utility function is

$$V(E,p) = \Sigma\beta_i ln\beta_i + ln(E - p'\gamma) - \Sigma\beta_i ln p_i, \tag{5}$$

and the closed-loop E.L.E.S. consumption function is

$$E = \delta(W - \frac{p'\gamma}{\rho}) + p'\gamma \tag{6}$$

giving, by Roy's Identity, the E.L.E.S. system

$$p_i q_i = p_i \gamma_i + \mu \beta_i (z - p'\gamma) \text{ with } \mu = \frac{\delta}{\rho}, \quad z = \rho W .$$

The role of the perfect financial capital market has been to remove the dependence of the time path of $q(t)$ on the time path of $y(t)$. In the extreme imperfect capital market case, we would have $E(t) = y(t)$, and there are various alternatives of partial dependence in between.

III. The Production Sector

In section II, the assumption of a perfect capital market with interest rate ρ allowed concentration on the consumption problem, given the maximized present value of the consumer. It was found that the use of duality theory allowed considerable simplification by separating the static optimization problem from the intertemporal optimization. Similar separation and simplification is possible in the production sector, and we will now move straight to the appropriate stacking of optimization problems and the associated dual specifications.

On the production side, consider the following structure:

k capital goods

$$K' = K_1, \ldots, K_k$$

with corresponding purchase prices

$$c' = c_1, \ldots, c_k ;$$

ℓ variable inputs

$$L' = L_1, \ldots, L_\ell$$

with wage rates

$$w' = w_1, \ldots, w_\ell ;$$

and m outputs

$$Q' = Q_1, \ldots, Q_m$$

with selling prices

$$s' = s_1, \ldots, s_m .$$

Production technology is given by the production possibility set $T = \{Q, K, L\}$, a point set of feasible input-output combinations. It is often useful to assume a frontier of the form $f(Q, K, L) = 0$ which in the case of $m = 1$ is the usual production function. For our purposes, the distinction between capital goods and variable inputs is that capital goods are owned by the enterprise, requiring commitment of financial resources, whereas variable inputs are paid for out of current revenue. It will become clear that with an imperfect capital market new capital goods compete directly with consumption, whereas variable inputs do not.

Duality theory considers the problem of choosing output and variable inputs, given fixed inputs, in order to maximize net revenue. Then the variable profit or net revenue function is defined by

$$NR(K,w,s) \;=\; \max_{Q,L} \left\{ s'Q - w'L \;:\; (Q,K,L)\epsilon T \right\}. \tag{7}$$

There is a duality among $NR(K,w,s)$, T and the cost function $C(K,Q,w)$, each being an equivalent representation of the underlying production technology. But Hotelling's Lemma gives

$$QNR_i = \frac{\partial NR}{\partial s_i} \quad \text{(supply of } Q_i\text{)}, \quad i=1, \ldots, m,$$

$$LNR_i = \frac{\partial NR}{\partial w_i} \text{ (demand for } L_i\text{)}, \quad i=1, \ldots, \ell,$$

so that it is obviously more convenient to specify NR than T or C.

In the longer run there is the possibility of re-allocating total capital investment $X = c'K$ into alternative configurations of fixed capital assets. Thus given X consider the revenue function defined by the problem of choosing K to maximize net revenue subject to $c'K = X$, *i.e.* define

$$\begin{aligned} R(X,w,s,c) &= \max_{K} \left\{ NR(K,w,s) \;:\; c'K = X \right\} \\ &= \max_{K,L,Q} \left\{ s'Q - w'L \;:\; (Q,L,K)\epsilon T,\; c'K = X \right\} \end{aligned} \tag{8}$$

This leads to the capital input demand functions based on R,

$$KR(X,w,s,c)$$

as well as the variable input demand functions and output supply functions based on R,

$$\begin{aligned} LR(X,w,s,c) &= LNR(KR(X,w,s,c),w,s) \\ QR(X,w,s,c) &= QNR(KR(X,w,s,c),w,s). \end{aligned}$$

Although optimization problem (8) is not covered explicitly by the existing duality literature, existing results can be easily extended to show that T, f, C, NR and R are all dually related, and equivalently represent the underlying production technology. The result analogous to Roy's Identity, Shephard's Lemma and Hotelling's Lemma is:

$$KR_i \;=\; -\frac{\partial R}{\partial c_i} \bigg/ \frac{\partial R}{\partial X} \tag{9}$$

$$LR_i \;=\; \frac{\partial R}{\partial w_i} \tag{10}$$

$$QR_i \;=\; \frac{\partial R}{\partial s_i} \tag{11}$$

(9) is formally analogous to Roy's Identity. Consider (10).

Now $$\frac{\partial R}{\partial w_i} \;=\; \sum_j \frac{\partial NR}{\partial K_j} \; \frac{\partial K_j}{\partial w_i} + \frac{\partial NR}{\partial w_i}$$

$$= \sum_j \lambda c_j \frac{\partial K_j}{\partial w_i} + L_i \quad \text{(by optimality and Hotelling's Lemma)}$$

and $$\sum_j c_j \frac{\partial K_j}{\partial w_i} = 0 \text{ since } c'K = X\,.$$

From the point of view of an integrated consumption-production-investment-financing model, the function R is the appropriate specification of the productive sector of the model, and $\frac{\partial R}{\partial X}$ represents the marginal return of \$1 invested in the production sector. In the literature, attention is concentrated on functional forms for f, C, or NR. Thus functional forms for R will have to be derived from these, or specified directly subject to appropriate conditions (*e.g.* increasing in s, decreasing in c, w, increasing in X, homogeneous of degree 1 in s, w, homogeneous of degree 0 in c, X, plus appropriate curvature restrictions).

IV. THE FINANCING PROBLEM

Define the net worth of the individual at any point in time by $W(t)$. Let $B(t)$ represent total borrowing at time t. Then in the absence of a perfect financial capital market $W(t)$ is defined by

$$W(t) = X(t) - B(t)$$

The imperfection in the financial capital market is represented by the average rate of interest schedule which is taken to be an increasing function of the debt-asset ratio; *i.e.* let $Z = \frac{B}{X}$, and specify the average rate of interest on B by $r(Z)$, $r'(Z) > 0$, $Z > 0$. At any point in time the flow of funds constraint is

$$R + \dot{B} = E + \dot{X} + r(Z) . B \tag{12}$$

Equation (12 represents the way in which production, consumption, financing and investment decisions interact.

V. THE JOINT FIRM-HOUSEHOLD MODEL

In sections II and III we have suggested that in the context of an intertemporal model, the appropriate specification of the consumption sector is by means of the indirect utility function $V(E,p)$, with $E(t)$ the single intertemporal decision variable, and the appropriate specification of the productive sector is by means of the revenue function $R(X,w,s,c)$, with $X(t)$ the single intertemporal decision variable. The two components are tied together by the cost of funds schedule $r(Z)$. If $r'(Z) \equiv 0$ Fisher's separation result would apply and the two stage procedure alluded to in the introduction would apply. Provided R had diminishing returns to scale in X there would be an optimal scale of X, say X^0, and $X(0)$ would be immediately adjusted to X^0, with the present value of $R(X^0,w,s,c)$ available for allocation to consumption as in section II. If on the other hand borrowing was unavailable, a model such as McLaren [18] would apply. In this section we consider the case of a monotonically increasing $r(Z)$ function, perhaps the most realistic case.

Based on the specifications of sections II, III and IV we now consider the following intertemporal optimization model:

$$\text{maximize} \int_0^\infty e^{-\delta t} V(E,p)\,dt \tag{13}$$

$$\begin{aligned} \text{subject to} \quad & X(t) - B(t) = W(t) \\ & \dot{W} \;\;= R(X,w,s,c) - E - r(Z)B \\ & W(0) = W^0 = X^0 - B^0 \;, \\ & \lim W(t) \geqslant 0 \,, \\ \text{where} \quad & Z = B/X \text{ and } r'(Z) > 0 \;. \end{aligned}$$

Structure to the problem so far is given by the intertemporal additivity of the utility functional and the additive structure of the accounting identity. Further structure will be given by specifying functional forms for V, R and r, but the major behavioural aspects of the model are contained in the specification above. Briefly, utility is generated only by consumption spending, so that, for example, a large farm is not desirable *per se*, but only in so far as it generates revenue to allow purchase of consumption goods. Prices of the consumption goods, p, outputs, s, variable inputs, w, and the capital goods, c, are assumed constant in planning time, independent of decisions of the enterprise. The enterprise borrows from external sources, and pays an average rate of interest given by the $r(Z)$ schedule. Therefore although the actual rate of interest is determined by the actions of the enterprise, and so endogenous, the schedule itself is subject to the same assumptions as prices. Thus strictly speaking the model will generate demand schedules for funds, capital goods, consumer goods and variable inputs, and supply schedules for outputs, and so is only part of a general equilibrium specification. Such a complete model would jointly determine all prices and quantities. Such a general equilibrium approach is not followed here, since the model we have in mind would be used for a data base consisting of panel data on individual enterprises, and so an assumption of constant prices may be reasonably appropriate, allowing treatment of demand and supply schedules as reduced forms.

One problem generated by the assumption of constant prices is the implied perfect second-hand market for capital goods, clearly an unrealistic assumption. For example, a well cannot be sold. This problem can be partially offset by appropriate specification of the production function, so that instead of specifying land, fences and wells as separate capital goods, they would be lumped together as "improved land".

In the basic model there are two sources of funds, net revenue from productive activities and net new borrowing, and three uses of funds, consumption spending, net investment and interest payments. A maintained assumption is that all enterprises are net borrowers, and this is assumed in the specification of $r(Z)$. While it would be relatively easy to introduce the possibility of lending funds at a constant rate of interest, this can be subsumed in the R function, which has already determined the best use of funds. The assumption that the average rate of interest increases with the debt-asset ratio is somewhat arbitrary, but is probably not too unrealistic. To make the basic model operational, refinements such as taxes, tax credits, depreciation and external income sources would be introduced.

The model has one state variable, W , and four controls E,B,X,Z, of which two are redundant. In the solution E and B are treated as controls, with X and Z retained simply as definitions. In this section dependence on prices is suppressed. Introduce the Hamiltonian:

$$H(t,W;E,B;\mu) = e^{-\delta t}V(E) + \mu(R(X) - E - r(Z)B)\,. \tag{14}$$

Then the first-order necessary conditions are:

$$\text{(i)} \qquad \dot{W} = H_\mu = R(X) - E - r(Z)B$$

$$\text{(ii)} \qquad \dot{\mu} = -H_W = -\mu(R'(X) + Z^2 r'(Z))$$

$$\text{(iii)} \qquad H_E = e^{-\delta t}V_E - \mu = 0$$

$$\text{(iv)} \qquad H_B = \mu(R'(X) - r(Z) - r'(Z)Z(1-Z)) = 0$$

Condition (iv) allows solution for the optimal debt asset ration, Z^*, and therefore for optimal B and X, in terms of W.

This condition can be interpreted as the equation of the marginal return from borrowing $R'(X)$, with the marginal cost of borrowing, made up of the direct cost $r(Z)$ plus the marginal effect on interest rates of increased borrowing. Condition (iii) allows solution for E in terms of μ , and so the necessary conditions (i) to (iv) can be reduced to two differential equations in W and μ . To say much more requires particular specifications.

To illustrate a particular solution, consider the convenient, and perhaps important, case in which R is linear in X (equivalent to constant returns to scale in f). Then

$$R(X,w,s,c) = h(w,s,c)\,.\,X$$

and (14) (iv) gives $h = r(Z) + r'(Z)Z(1-Z)$, so that the resulting optimal Z , say Z^*, is independent of W and hence constant in planning time. Now define

$$g = h + r'(Z^*)Z^{*2} = r(Z^*) + r'(Z^*)Z^*$$

so

$$\dot{\mu} = -g\mu, \mu(t) = \mu_0 e^{-gt} \text{ with } \mu_0 \text{ a constant.}$$

Equation (14) (iii) then allows solution of $E(t)$ in terms of μ_0 and h , the explicit solution depending on the functional form chosen for V . To continue the illustration, specify V as the indirect utility function corresponding to a Klein-Rubin direct utility function, equation (5). Then condition (14) (iii) gives

$$\frac{e^{-\delta t}}{E - p'\gamma} = \mu_0 \; e^{-gt}$$

or

$$E = p'\gamma + \frac{1}{\mu_0} e^{(g-\delta)t}$$

From this point on, only optimal Z appears, so the * is deleted. Now

$$\begin{aligned}\dot{W} &= hX - E - r(Z)B \\ &= \frac{h}{1-Z} \cdot W - E - r(Z)\,\frac{Z}{1-Z}\,W\end{aligned}$$

But
$$\begin{aligned}h - r(Z)Z &= r(Z) + r'(Z)Z - r'(Z)Z^2 - r(Z)Z \\ &= (1-Z)\,[r(Z) + r'(Z)Z] \\ &= (1-Z)g\end{aligned}$$

so
$$\dot{W} = gW - \frac{1}{\mu_0}\,e^{(g-\delta)t} - p'\gamma$$

with general solution
$$W(t) = \frac{e^{(g-\delta)t}}{\mu_0\delta} + \frac{p'\gamma}{g} + C_1 e^{gt}.$$

Transversality gives $C_1 = 0$ and $\frac{1}{\mu_0} = \delta\,[W(0) - \frac{p'\gamma}{g}]$ giving the open-loop solution for $W(t)$, $\mu(t)$ and $E(t)$. In fact
$$\frac{1}{\mu(t)} = \delta\,(W(t) - \frac{p'\gamma}{g})e^{\delta t}$$

giving the closed-loop solution for $E(t)$
$$E = p'\gamma + \delta\,(W - \frac{p'\gamma}{g}). \tag{15}$$

The optimal closed-loop solutions for B and X are simply stated in terms of Z :
$$X = \frac{1}{1-Z}\,W \tag{16}$$
$$B = \frac{Z}{1-Z}\,W \tag{17}$$

and similarly for $\dot{X}$ and $\dot{B}$, using
$$\dot{W} = (g-\delta)\,[W - \frac{p'\gamma}{g}]\ ;$$
$$\dot{X} = \frac{1}{1-Z} \cdot (g-\delta)\,[W - \frac{p'\gamma}{g}] \tag{18}$$
$$\dot{B} = \frac{Z}{1-Z}\,(g-\delta)\,[W - \frac{p'\gamma}{g}] \tag{19}$$

Equation (15) should be compared with (6) while recalling the new definition of W .

VI. BEHAVIOUR IN CALENDAR TIME

Equations (15), (16) and (17) represent the closed loop solution for the dynamic component of the model, giving controls at each point in time as a function of the state at that time, W , and parameters of the system including prices. Recall that Z is independent of W provided R is linear in X . One may be tempted to substitute (18) and (19) for

(16) and (17), thus treating investment and net borrowing as the endogenous variables. However, (18) and (19) would not be useful for empirical work. Recall that the initial condition is $W(0) = W^0 = X^0 - B^0$. While $W(t)$ is continuous at $t = 0$, there is no constraint that $X(0) = X^0$, $B(0) = B^0$; in fact the restriction is $X(0) - B(0) = X^0 - B^0$. Thus in planning time an initial decision is made to bring B and X into an optimal relationship and from then on (18) and (19) would generate optimal paths in planning time. But in actual time, a change in any parameters that affect $Z\ (r(Z), w, s, c)$ will require a re-alignment of B and X . Thus an "appropriate" investment or net borrowing equation must account for the continual re-alignment of B and X as parameters change, and can be derived by totally differentiating the closed-loop solutions with respect to t . Now

$$\frac{dX}{dt} = \frac{\partial X}{\partial W}\frac{dW}{dt} + \frac{\partial X}{\partial Z}\frac{dZ}{dt}$$

and
$$\frac{dW}{dt} = \dot{W} + \left(\frac{dc}{dt}\right)' K$$

(where a . indicates a time derivative along an optimal path). Therefore

$$\frac{dX}{dt} = \frac{(g-\delta)}{1-Z}\left[W - \frac{p'\gamma}{g}\right] + \frac{1}{1-Z}\left(\frac{dc}{dt}\right)' K$$
$$+ \frac{W}{(1-Z)^2}\left(\frac{\partial Z}{\partial c}\frac{dc}{dt} + \frac{\partial Z}{\partial w}\frac{dw}{dt} + \frac{\partial Z}{\partial s}\frac{ds}{dt}\right) \tag{20}$$

which decomposes investment into its three components: movement along an optimal path, movement to a new optimal path due to capital gains, and refinancing due to a change in the relation between cost of borrowing and return on borrowing. While relations such as (20) may be useful for analysis, empirically it would seem preferable to use (15), (16) and (17). It should be mentioned that had durable goods been included, equation (15) for E would have to be similarly modified. (This generalization for E.L.E.S. has been carried out by Dixon and Lluch [5] and could easily be integrated into the model through the consumption module.)

VII. SOME SUGGESTIONS FOR IMPLEMENTATION

The functional form assumed for $V(E,p)$ and the restriction on the form of $R(X,w,s,c)$ in section V allowed explicit solution of the necessary conditions. Empirical implimentation would require functional forms for R and $r(Z)$, and we now turn to some possible examples.

For the revenue function R , one approach would be to specify a form for $f(Q, K, L)$ and derive $R\ (X,w,s,c)$ by constrained maximization. This approach is feasible for a Cobb-Douglas or C.E.S.–C.E.T. specification, but while these are useful in that they parameterize on the elasticities of substitution and transformation, they are too restrictive for more than two inputs or two outputs. Generalization, for example to a CRETH-CRESH formulation, would not seem to enable an explicit derivation of R , although we should note that the method developed in Vincent, Dixon and Powell [24] may allow $\partial R/\partial X$ to be estimated as a shadow price, even if the explicit analytical form is unknown. Alternatively, deriving R from a specification of $NR\ (K,w,s)$ is a simpler constrained optimization problem. Thus consider Diewert's translog variable profit function (where $v = s, w$):

$$\begin{aligned} lnNR\,(K, v) = {} & \alpha_0 + \Sigma\alpha_i lnv_i + \tfrac{1}{2}\sum_i\sum_h \gamma_{ih}\, lnv_i\, lnv_h \\ & + \sum_i\sum_j \delta_{ij}\, lnv_i\, lnK_j + \sum_j \beta_j\, lnK_j \\ & + \tfrac{1}{2}\sum_j\sum_k \phi_{jk}\, lnK_j\, lnK_k\,, \end{aligned} \tag{21}$$

see Diewert [4].

The following constraints are sufficient to ensure homogeneity of degree 1 in v:

(i) $\sum_i \alpha_i = 1$,

(ii) $\sum_i \delta_{ij} = 0$ for all j,

(iii) $\gamma_{ih} = \gamma_{hi}$ for all i, h,

(iv) $\sum_i \gamma_{ih} = 0$ for all h.

Homogeneity of degree 1 in K requires

(v) $\sum_j \beta_j = 1$,

(vi) $\sum_j \delta_{ij} = 0$ for all i,

(vii) $\sum_j \phi_{jk} = 0$ for all k,

(viii) $\sum_k \phi_{jk} = 0$ for all j.

Solution for the form of R is facilitated by the further constraint

(ix) $\phi_{jk} = 0$ for all j, k

This constraint is justified only on the basis of mathematical tractability, and its implications are considered below. Using this constraint, and maximizing (24) subject to $c'K = X$ gives the set of conditions

$$\lambda c_j K_j = (\beta_j + \sum_i \delta_{ij}\, lnv_i)$$

and using conditions (v) and (vi) gives $\lambda = \frac{1}{X}$ so

$$\frac{c_j K_j}{X} = (\beta_j + \sum_i \delta_{ij}\, lnv_i) \tag{22}$$

Thus the proportion of X allocated to capital good j depends on prices of variable inputs and outputs. This means that the effect of the constraints $\phi_{jk} = 0$ is to constrain the elasticity of substitution between capital inputs to be unity. However the specification is more general than Cobb-Douglas in capital inputs, as the above proportion is not constant. Hotelling's Lemma gives the following share equations for variable inputs and outputs:

$$S_i = \alpha_i + \sum_h \gamma_{ih}\, lnv_h + \sum_j \delta_j\, lnK_j \tag{23}$$

Substitution of (25) into (24) gives lnR, and it can be seen that

$$\begin{aligned} ln \frac{\partial R}{\partial X} = {} & \alpha_0 + \sum_i \alpha_i\, lnv_i + ½ \sum_i \sum_h \gamma_{ih}\, lnv_i\, lnv_h - \sum_i \sum_j \delta_{ij}\, lnv_i\, lnc_j \\ & + \sum_i \sum_j \delta_{ij}\, lnv_i\, ln\,(\beta_j + \sum_k \delta_{kj}\, lnv_k) \\ & - \sum_j \beta_j\, lnc_j + \sum_j \beta_j\, ln\,(\beta_j + \sum_i \delta_{ij}\, lnv_i)\,. \end{aligned}$$

This equation gives h.

An even simpler procedure, consistent with the idea of duality between NR and R and with the restriction that $R\,(X,w,s,c) = h\,(w,s,c)\,X$ would be to use a flexible functional form for h, for example a translog specification.

Turning now to $r(Z)$, the simplest specification would seem to be

$$r(Z) = r_1 + r_2 Z$$

since at least two parameters seem necessary to capture the two aspects of "general level" and "effect of security". Then (14) (iv) gives

$$h = r_1 + 2r_2 Z - r_2 Z^2$$

and

$$Z = \frac{2r_2 \pm \sqrt{4r_2^2 - 4r_2\,(h - r_1)}}{2r_2}$$

Since we are interested in $0 < Z < 1$, the "minus" solution is clearly appropriate, giving the two boundaries for Z:

$$Z = 0 \text{ for } h = r$$

$$Z = 1 \text{ for } h = r_1 + r_2$$

Simplifying the solution for Z:

$$Z = 1 - \left(\frac{r_2 - h - r_1}{r_2}\right)^{½} \tag{24}$$

Clearly a full systems implementation would be desirable and fully efficient, but hardly practicable. A multi-step estimation procedure, based on the multi-stage nature of the optimization, may represent enough saving in computing to offset any inefficiency. For example, if we denote observations by O_{ijt}, meaning enterprise i, involved in activity j at time t, and assuming all activities and individuals face the same $r(Z)$ schedule, r_1 and r_2 can be estimated for each time point t, thus generating a time series of r_1, r_2. The stacking of optimization problems is: given W choose X, given X choose K, given K choose L and Q . Thus for each activity j , observations across time and individuals allow estimation of (22) and (23), subject to appropriate restrictions. Only α_0 remains, and this can be estimated by moving all variables to the left of (21). A time trend might be appropriate in (21) to allow for technical progress. Alternatively, (21), (22) and (23) may be estimated concurrently, or again (22) could be estimated first and a purged K series used in (23). At this stage the R function has been estimated for each activity j , and together with the estimated r_1 and r_2 series, can be used to generate a time series of Z from (27) for each activity j . Here inefficiency is apparent, since observations on Z , have not been utilized. From this series on Z, a time series (across categories) on g is constructed by

$$g = r_1 + 2r_2Z$$

This series could then be used in order to estimate (15), giving estimates of the parameters of the utility function, using precisely the methods commonly used in the estimation of E.L.E.S., *i.e.* full systems estimation.

VIII. CONCLUSION

In section II Lluch's E.L.E.S. was reformulated to demonstrate that the appropriate state variable is net worth of the consumer, and duality theory was used to allow an economical statement of the intertemporal problem. The production sector was similarly specified in section III where a new type of duality was suggested and the analogue of Roy's Identity and Hotelling's Lemma derived. These two specifications were then tied together by the assumption of a cost of funds schedule increasing in the debt-asset ratio, to give the joint firm-household model of section V, an integrated model of consumption, production, investment and financing decisions. In section VI we distinguished between movement along an optimal path and movement to a new path, a distinction that will be important to obtain correct specification for empirical work. Finally, some possible functional forms were suggested. The model is intended to be applied to time-series of cross-sections of observations on individual firm-households, a prime example being farms. We hope to report on such an application in future.

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